MICROCOPY CHART

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFIT/CI/NR 86-46T	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE *(and Subtitle)* Application of YBJ Control Theory to the Initial Response Problem		5. TYPE OF REPORT & PERIOD COVERED THESIS/~~DISSERTATION~~
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Paul Jay Warden		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS AFIT STUDENT AT: Arizona State University		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS AFIT/NR WPAFB OH 45433-6583		12. REPORT DATE 1986
		13. NUMBER OF PAGES 101
14. MONITORING AGENCY NAME & ADDRESS*(if different from Controlling Office)*		15. SECURITY CLASS. *(of this report)* UNCLASS
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT *(of this Report)*

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED

17. DISTRIBUTION STATEMENT *(of the abstract entered in Block 20, if different from Report)*

18. SUPPLEMENTARY NOTES
APPROVED FOR PUBLIC RELEASE: IAW AFR 190-1

LYNN E. WOLAVER   11 April 86
Dean for Research and
Professional Development
AFIT/NR, WPAFB OH 45433-6583

19. KEY WORDS *(Continue on reverse side if necessary and identify by block number)*

DTIC
ELECTE
APR 22 1986

A

20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

DD _{1 JAN 73} FORM 1473   EDITION OF 1 NOV 65 IS OBSOLETE

ABSTRACT

Two theorems were derived. First, the <u>Initial Response Theorem</u>
describes the necessary and sufficient conditions for a series compensator
for a feedback control system to simultaneously stabilize the system,
and cause the initial system response to achieve prescribed constraints.
A single input-single output, continuous time linear system is considered,
with a delta function driving any rational transfer function in the
s-domain being the system input. Design constraints can be placed on
the initial response value, and on any of the derivatives (from the right)
of the initial response. Second, the <u>Initial Response Parameterization</u>
gives a parameterization of the complete set of compensators that will
meet the given constraints whenever the conditions of the Initial
Response Theorem are met. The area of Youla, Bongiorno and Jabr (YBJ)
Control Theory was used for the derivations, but first required a
system transformation to convert the initial value problem to one of
stability.

APPLICATION OF YBJ CONTROL

THEORY TO THE INITIAL

RESPONSE PROBLEM

by

Paul Jay Warden

A Thesis Presented in Partial Fulfillment
of the Requirements for the Degree
Master of Science

ARIZONA STATE UNIVERSITY

May 1986

APPLICATION OF YBJ CONTROL
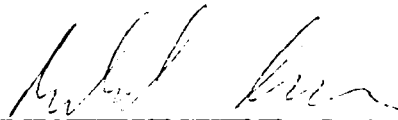
THEORY TO THE INITIAL

RESPONSE PROBLEM

by

Paul Jay Warden
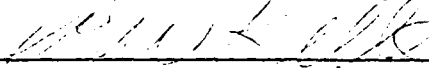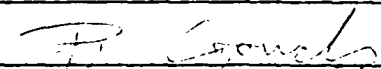
has been approved

December 1985

APPROVED:
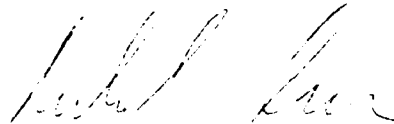
_____ ,Chairperson

_____

_____
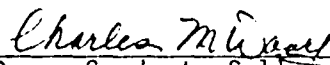
Supervisory Committee

ACCEPTED:

_____
Department Chairperson

_____
Dean, Graduate College

ACKNOWLEDGEMENTS

This thesis would not have been possible without the help of Dr. Richard Saeks, my committee chairman. He introduced me to the YBJ Control Theory, and devoted many hours of his valuable time researching the material within this thesis with me. I would like to thank Dr. Saeks for all his help, and for the opportunity to learn from him.

I would also like to thank the other members of my committee, Dr. Peter Crouch and Dr. Richard Kelly, for their much appreciated guidance, and my friend Kim, for her hard work while typing this rather lengthy document.

Finally, I would like to voice my appreciation to the United States Air Force for providing me with the opportunity to attend Arizona State University full time to work on this master's degree.

TABLE OF CONTENTS

Page

LIST OF FIGURES

Page

Figure

CHAPTER I

INTRODUCTION

1.1 Objective

The results of this thesis are two theorems. The first is named
the _Initial Response Theorem_ (IRT), which describes the necessary and
sufficient (N & S) conditions for a series compensator for a feedback
control system to simultaneously stabilize the system, and cause the
initial system response to achieve prescribed constraints. A single
input-single output, continuous time linear system is considered. The
system can have as its input a delta function driving any rational
transfer function in the s-domain. Design constraints can be placed
on the initial response value, and on any of the derivatives of the
initial response, i.e., as time approaches 0^+ (0 from the positive
side). The second theorem is termed the _Initial Response Parameterization_
(IRP), and it gives a parameterization of the complete set of compensators
that will meet the given constraints whenever the conditions of the
IRT are met.

The motivation for this work was the intuition that the shape of
the transient response can be controlled by controlling the initial
response. Thus, if the initial response and the asymptotic stability
can be simultaneously controlled, the designer will have a tool that
can help meet both transient response requirements and stability
requirements simultaneously. Examples of this technique are given in
Chapter IV.

The theorems are derived in essentially two stages. First, the

given system is modified to another system. Any compensator causing the modified system to be stable will also cause the original system to meet the desired constraints. Then, the relatively new area of YBJ control theory, which is introduced in Section 1.3, is applied to the modified system. This leads to the N & S conditions, and the parameterization describing the complete set of compensators that will stabilize the modified system, and therefore also meet the constraints given for the original system. The constraint of system stability is achieved by application of previous results in YBJ (10). the initial condition constraints are met via the application of new results in YBJ derived in this paper. All of the results assume that the plant considered is a proper rational function.

1.2 Example
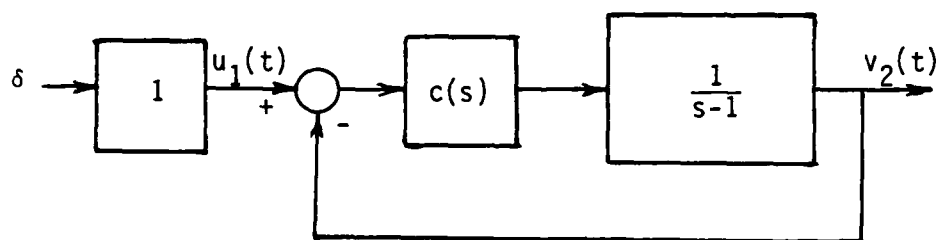
Figure 1. Example System

As an example, consider the system shown in Figure 1. The input is an impulse, which is represented by a delta function driving a transfer function of value 1. The plant has a transfer function

$$p(s) = \frac{1}{(s-1)} \tag{1.1}$$

and is unstable. It is desired that the system $H_{v_2 u_1}(s)$ be stable, and that the initial response $v_2(0^+)$ equal 3.

The parameterization of the complete set of compensators that stabilize the feedback loop has been found before this thesis (10). Without going into further detail at this point, it will suffice to state that this parameterization leads to $c(s)$ of the form

$$c(s) = \frac{-w(s) \frac{(s-1)}{(s+1)} + 2}{w(s) \frac{1}{(s+1)} + 1} \qquad \langle 1.2 \rangle$$

where $w(s)$ is any stable function.

The compensator must also meet the constraint $v_2(0^+)$ equal 3. The system has the transfer function

$$H(s) = 1 \cdot \frac{c(s)p(s)}{1 + c(s)p(s)} \qquad \langle 1.3 \rangle$$

If $\langle 1.1 \rangle$ and $\langle 1.2 \rangle$ are substituted into Equation 1.3, then after some manipulation

$$H(s) = -w(s) \frac{s-1}{(s+1)^2} + \frac{2}{s+1} \qquad \langle 1.4 \rangle$$

Also, according to the Initial Value Theorem

$$\lim_{s \to \infty} \{sH(s)\} = \lim_{t \to 0^+} \{h(t)\} \qquad \langle 1.5 \rangle$$

where $H(s)$ is the Laplace Transform of $h(t)$. Thus, it is also required that

$$\lim_{s \to \infty} \left\{-w(s) \frac{s(s-1)}{(s+1)^2} + \frac{2s}{s+1}\right\} = 3 \qquad \langle 1.6 \rangle$$

in order to meet the initial response constraint. One could then search

"brute force" for stable w(s) which satisfy Equation 1.6 in order to solve the problem, and could achieve particular solutions such as

$$w(s) = \frac{-s^2}{(s+1)^2} \tag{1.7}$$

However, the IRP in Chapter III yields the complete set of compensators. The calculation is not difficult for this problem, and will be omitted here to keep the example brief. But, when applied, the IRP yields

$$w(s) = \frac{-s^2 -3s}{(s+1)^2} + \frac{1}{s+1} e(s) \tag{1.8}$$

as the complete parameterization, where e(s) is arbitrarily stable, and c(s) is again described in Equation 1.2.

1.3 YBJ Control Theory

The foundation of YBJ control theory is two papers published in 1976 by Youla, Bongiorno, and Jabr (16, 17). A result of their work was the complete parameterization of the set of stablizing compensators for a multivariate feedback system, based on a new approach in feedback system design. Since then, this new approach has led to similar parameterizations of other problems, such as the tracking and disturbance rejection problems (9, 10, 11), and also to results in optimization theory based on the parameterizations found (16, 17).

The basic approach to discovering the parameterizations (other than the original stability parameterization) has been to arrange the mathematics so that they, too, were a stability problem. Thus similar, though different approaches to the original stabilization problem could

be used. This then is the reason for the system transformation used
in the derivation of the theorems in this paper, as will be seen in
the following chapters.

The key to the YBJ approach is to use a "stable fractional
representation" for the transfer functions considered. This is
opposed to the "polynomial fractional representation" used classically.
For example, the function r(s) would classically be written in the form

$$r(s) = \frac{p(s)}{q(s)} \qquad \qquad <1.9>$$

where p(s) and q(s) are polynomials. But, YBJ requires that r(s) be
written

$$r(s) = \frac{n_r(s)}{d_r(s)} \qquad \qquad <1.10>$$

where $n_r(s)$ and $d_r(s)$ are both stable, and have no common (closed) right
half-plane (RHP) zeros, including infinity. This is termed "RHP
coprimeness", or simply "coprimeness". Thus, if m(s) is a Hurwitz
polynomial of order equal to the order of r(s), then

$$n_r(s) = \frac{p(s)}{m(s)} \qquad \qquad <1.11>$$

and

$$d_r(s) = \frac{q(s)}{m(s)} \qquad \qquad <1.12>$$

are possible representations for YBJ. Then, in the case that $n_r(s)$
and $d_r(s)$ are RHP coprime, YBJ dictates that there must exist stable
$u_r(s)$ and $v_r(s)$ such that

$$u_r(s)n_r(s) + v_r(s)d_r(s) = 1 \qquad \qquad <1.13>$$

This new representation is the basis of the YBJ control theory. For further introduction to YBJ, the reader is referred to (10) which is drawn upon extensively in this paper, and to (9, 11, 16, 17, 18).

1.4 Summary of Results
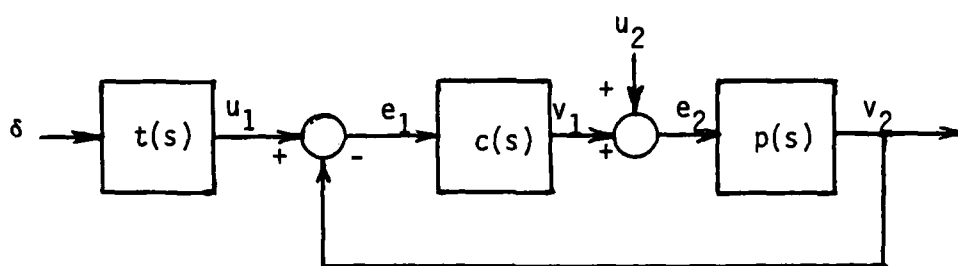
Figure 2. General System Configuration

For the system in Figure 2, assume that $t(s)$ represents a given input to a feedback system $H_{v_2 u_1}(s)$. Also, assume that the finite (real numbers) constraints $\{y_0, y_1, \ldots, y_j\}$ are given on the initial time response of the system $\{v_2^0(0^+), v_2^1(0^+), \ldots, v_2^j(0^+)\}$, where the superscript represents the n-th derivative, and assume that the system $H_{v_2 u_1}(s)$ is to be stabilized. Then define M to be the lowest order initial response derivative specified that is non-zero, and define j as the highest order derivative specified, so that $j \geq M$. Also, define $r\{H(s)\}$ to be the "relative degree" of a transfer function $H(s)$, such that

$$r\{H(s)\} = (\#\text{finite poles } H(s) - (\#\text{finite zeros } H(s)) \qquad <1.14>$$

Next define

$$p(s) = \frac{n_p(s)}{d_p(s)} \qquad \qquad <1.15>$$

and

$$t_n(s) = s^{n+2} \, t(s) = \frac{n_{t_n}(s)}{d_{t_n}(s)} \qquad \qquad <1.16>$$

where $n_p(s)$ is stable and coprime with stable $d_p(s)$, and where $n_{t_n}(s)$ is stable and coprime with stable $d_{t_n}(s)$.

With the above definitions in mind, the following theorom is a major result of this thesis.

Initial Response Theorem

For the system in Figure 2, assume that $r\{p(s)\} \geq 0$. Also assume that

The input $t(s)$ has fewer than three poles at $s = 0$.   <1.17>

or

If the above condition is not met,

then $\{(\#poles \; t(s) \; at \; s=0)-3\} < \{\#zeros \; p(s) \; at \; s=0\}$.   <1.18>

Then, a set of compensators $c(s)$ exists that will simultaneously stabilize $H_{v_2 u_1}(s)$ (without RHP pole-zero cancellations between $p(s)$ and $c(s)$), and meet the initial condition constraints $\{v_2^0(0^+), \ldots, v_2^j(0^+)\} = \{y_0, \ldots, y_j\}$ if and only if (iff)

$$r\{t(s)\} + r\{p(s)\} \leq M+1 \qquad \qquad <1.19>$$

and

$$d_{t_j}(s) \text{ and } d_p(s) \text{ are RHP coprime,} \qquad <1.20>$$

which can be written $(d_{t_j}(s), d_p(s)) = 1$.

EOT (End of Theorem)

The proof for this theorem, as well as the resulting parameterization for the complete set of adequate compensators are given in Chapter III. The conditions given in equations <1.19> and <1.20> are not at all obvious. In practice, <1.20> should not normally pose a problem, while <1.19> will often be the more restrictive condition.

Though a different approach will be used in Chapter III to prove <1.19>, some intuition can be given (unrigorously) for its necessity now. It can be shown that the system $H_{v_2 u_1}(s)$ has a relative degree equal to or greater than that of the plant $p(s)$, whenever $r\{p(s)\} \geq 0$. Therefore, $v_2(s)$ must have relative degree

$$r\{v_2(s)\} \geq r\{t(s)\} + r\{p(s)\} \qquad <1.21>$$

since the relative degree of the product is the sum of the relative degrees. Also, the Initial Value Theorem (IVT) can be used to prove that

$$r\{v_2(s)\} = M+1 \qquad <1.22>$$

Thus, equating <1.21> and <1.22> leads to <1.19>.

The assumptions made in equations 1.17 and 1.18 are used to guarantee that a compensator meeting the n-th derivative constraint will also meet the 0-th through (n-1)th constraints. However, they may not be necessary conditions. At this time, a set of necessary and sufficient conditions have not been found, though the given assumptions are quite unrestrictive. If they do pose a problem, the interested

reader is referred to Chapter III for further details. Such a "guarantee" will be useful when applying the parameterization to be derived later.

The remainder of this thesis is organized as follows. Chapter II develops the system transformation, and develops a list of properties associated with the transformation. Chapter III applies the YBJ theory to the transformed system to discover the Initial Response Theorem and the resulting parameterization. Next, Chapter IV contains design examples using this theory. Finally, Chapter V contains a list of recommendations for further research on this topic.

CHAPTER II

SYSTEM TRANSFORMATION

2.1 Transformation Description

This chapter deals with the development of a system transformation
that modifies a given feedback system model so that YBJ control theory
can be applied in the next chapter. The transformation allows for
simultaneous system stability and initial response constraints to be
attacked with YBJ. The system is single input-single output continuous
time, and linear. The system input is a delta function driving a
rational transfer function.

As was discussed in the introduction, the approach to deriving
parameterizations for compensators using YBJ in the past has been to
look at each problem as one in stability. In some way, the algebra of
the problem was arranged so that stabilizing some quantity would produce
the desired results. Then, the YBJ theory could be used to find the
parameterization of compensators that would indeed stabilize the quantity
in question. Thus, the intent here is to derive a general system
transformation that leads to particular quantities, which if stabilized,
will cause the constraints to be met.

The problem at hand may have many constraints, but essentially
only two types. First, the feedback system must be stable. Second,
some initial value constraints must be met. Therefore, it is necessary
to derive some transformation that produces two quantities. The first
will stabilize the feedback loop when stabilized, and the second will
meet a desired initial condition constraint when stabilized. Thus, the

basic transformation concepts to be used here are as follows. First, the transformation of the feedback loop preserves the properties of stability, so that stabilizing compensators in the transformed domain also stabilize in the original s-domain. Because of this property, the stability constraint can be met using a previously developed stability parameterization. Second, the transformation yields a system output that meets one of the initial conditions when stabilized. Also, the transformation must have an inverse transformation, so that the answer can be of use in the original s-domain.

2.2 Mathematical Transformation

The first step in developing the system transformation is to develop a mathematical transformation that maps an initial derivative response into the final value response in a new domain. For this transformation let t and s represent the time and Laplace domain variables of a transfer function as usual. Then, let $\underline{t}$ and $\underline{s}$ represent the corresponding time and Laplace domain variables in the transformed domain. As such, the following lemma describes the mathematical transformation that performs the desired mapping.

Lemma 2.1

Let $H^n(s)$ be the n-th derivative of a transfer function $H(s) = L\{h(t)\}$. If $H^n(s)$ is strictly proper, and if $\overline{H}_n(\underline{s})$ only has poles in the left half plane, except possibly for a simple pole at s=0, then

$$\lim_{\underline{s} \to 0} \{\underline{s}\overline{H}_n(\underline{s})\} = \lim_{\underline{t} \to \infty} \{\overline{h}_n(\underline{t})\} \qquad \qquad <2.1>$$

$$= \lim_{t \to 0} \{h^n(t)\} = \lim_{s \to \infty} \{sH^n(s)\}$$

where $\overline{H}_n(\underline{s}) = \dfrac{1}{\underline{s}^{n+2}} H(\dfrac{1}{\underline{s}}) - \sum_{i=0}^{n-1} \dfrac{1}{\underline{s}^{n-i+1}} h^i(0^+) \qquad <2.2>$

and

$$\underline{s} = \frac{1}{s} \qquad \qquad <2.3>$$

Proof

The n-th derivative of a transfer function $H(s)$ is equal to

$$H^n(s) = s^n H(s) - \sum_{i=0}^{n-1} s^{n-1-i} h^i(0^+) \qquad \qquad <2.4>$$

$$n > 0$$

where $h^i(0^+) = \lim_{t \to 0^+} \{L^{-1}\{H^i(s)\}\} \qquad \qquad <2.5>$

The Initial Value Theorem (IVT) states that if $H(s)$ is strictly proper,

$$\lim_{t \to 0^+} \{h(t)\} = \lim_{s \to \infty} \{sH(s)\} \qquad \qquad <2.6>$$

Of course if $H(s)$ is not strictly proper, no finite initial value exists since there is an impulse at $t = 0$. Then, combining <2.4> and <2.6> leads to

$$\lim_{t \to 0^+} \{h^n(t)\} = \lim_{s \to \infty} \{sH^n(s)\}$$

$$= \lim_{s \to \infty} \{s^{n+1} H(s) - \sum_{i=0}^{n-1} s^{n-i} h^i(0^+) \qquad <2.7>$$

if $H^n(s)$ is strictly proper.

Next, define the transfer function

$$\overline{H}_n(\underline{s}) = L\{\overline{h}_n(\underline{t})\} \qquad \qquad <2.8>$$

in another domain, with the given variables. Then, the Final Value

Theorem (FVT) states that if $\overline{H}_n(\underline{s})$ has no (closed) RHP poles, except

for possibly pole at s = 0 (see (8), p. 714), then

$$\lim_{\underline{t}\to\infty} \{h_n(\underline{t})\} = \lim_{\underline{s}\to 0} \{\underline{s}H_n(\underline{s})\} \qquad <2.9>$$

At this point, a change of variables is necessary to force

$$\lim_{\underline{t}\to\infty} \{\overline{h}_n(\underline{t})\} = \lim_{t\to 0^+} \{h^n(t)\} \qquad <2.10>$$

This link can be accomplished by forcing equality in the Laplace domain,

by forcing

$$\lim_{\underline{s}\to 0} \{\underline{s}\overline{H}_n(\underline{s})\} = \lim_{s\to\infty} \{sH^n(s)\} \qquad <2.11>$$

$$= \lim_{s\to\infty} \{s^{n+1}H(s) - \sum_{i=0}^{n-1} s^{n-i}h^i(0^+)\}$$

If $\underline{s}$ and s are related by

$$\underline{s} = \frac{1}{s} \qquad <2.12>$$

then

$$\lim_{s\to\infty} \frac{1}{s} = 0 = \lim_{s\to 0} \underline{s} \qquad <2.13>$$

and

$$s^{n+1}H(s) - \sum_{i=0}^{n-1} s^{n-i}h^i(0^+) = \frac{1}{\underline{s}^{n+1}} H(\frac{1}{\underline{s}}) - \sum_{i=0}^{n-1} \frac{1}{\underline{s}^{n-i}} h^i(0^+) \qquad <2.14>$$

Thus, equating the values inside the brackets in Equation 2.10 dictates

that

$$\underline{s}\overline{H}_n(\underline{s}) = \frac{1}{\underline{s}^{n+1}} H(\frac{1}{\underline{s}}) - \sum_{i=0}^{n-1} \frac{1}{\underline{s}^{n-i}} h^i(0^+) \qquad <2.15>$$

which leads to

$$\overline{H}_n(\underline{s}) = \frac{1}{\underline{s}^{n+2}} H(\frac{1}{\underline{s}}) - \sum_{i=0}^{n-1} \frac{1}{\underline{s}^{n-i-1}} h^i(0^+) \qquad <2.16>$$

Therefore, invoking Equations 2.12 and 2.16 force the equality of Equation 2.11, and therefore also Equation 2.10, if the restrictions described for the IVT and the FVT are met.

EOP (End of Proof)

Lemma 2.1 then is a mathematical transformation yielding a new transfer function whose asymptotic value is equal to the initial n-th derivative of the original transfer function response. Later in this chapter a method of applying this to the feedback system shown in Figure 2 will be developed so that the YBJ theory can be used to meet the n-th derivative initial condition constraint. Also, Section 2.5 will prove that the restrictions on the IVT and the FVT will always be met in this application.

2.3 Inversion Transform and Properties

Before applying the transformation defined in Section 2.2 to a feedback system, it will prove useful to first define the transformation $\tilde{T}$ and its inverse $\tilde{T}^{-1}$ as follows.

$$\tilde{H}(\underline{s}) = \tilde{T} \{H(s)\} = H(\frac{1}{\underline{s}}) \qquad <2.17>$$

$$\tilde{T}^{-1} \{\tilde{H}(\underline{s})\} = \tilde{H} (\frac{1}{s})$$

The following properties associated with this transformation will also be useful. The proofs for these properties can be found in Appendix A.

Additive Property

$$\tilde{H}(\underline{s}) + \tilde{F}(\underline{s}) = \tilde{T} \{H(s) + F(s)\} \qquad <2.18>$$

Note that this property also implies the subtractive property.

Multiplicative Property

$$\tilde{H}(\underline{s}) \cdot \tilde{F}(\underline{s}) = \tilde{T}\ \{H(s) \cdot F(s)\} \qquad <2.19>$$

Note that this also implies division.

Inverse Transform Property

$$\tilde{H}(\frac{1}{\underline{s}}) = H(s) \qquad <2.20>$$

Transformation Stability Property

If $H(s)$ is stable (or unstable) in the s-domain, then so is $\tilde{H}(\underline{s})$ in
the $\underline{s}$-domain, and visa versa.

2.4 System Transformation

A method of using the results of Lemma 1 to transform the system
in Figure 2 (reproduced in Figure 3, part a) will be derived. This
will result in the n-th derivative of the initial ouput of the system

$$\lim_{t \to 0^+} \{v_2^n(t)\} \qquad <2.25>$$

to become the asymptotic ouput (time approaches infinity) of the
transformed system. This model will then be modified so that the YBJ
stability criterion may be used to meet the condtion in Equation 2.25.

Define the transfer functions

$$t_n(s) = s^{n+2} t(s) \qquad <2.26>$$

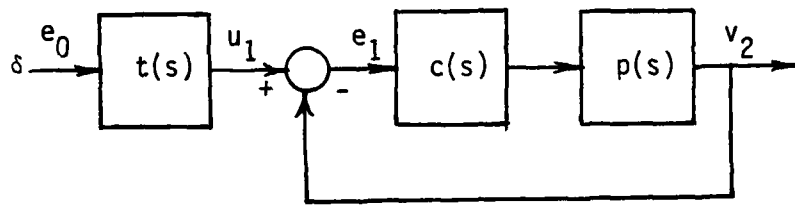$$\tilde{t}_n(\underline{s}) = \frac{1}{\underline{s}^{n+2}}\ \tilde{t}(\underline{s})$$

$$r_n'(s) = \sum_{i=0}^{n-1} s^{n-i+1} v_2^i(0^+) \qquad <2.27>$$

$$\tilde{r}_n'(\underline{s}) = \sum_{i=0}^{n-1} \frac{1}{\underline{s}^{n-i+1}}\ v_2^i(0^+)$$

Then Lemma 2.2 applies.

a. Given System
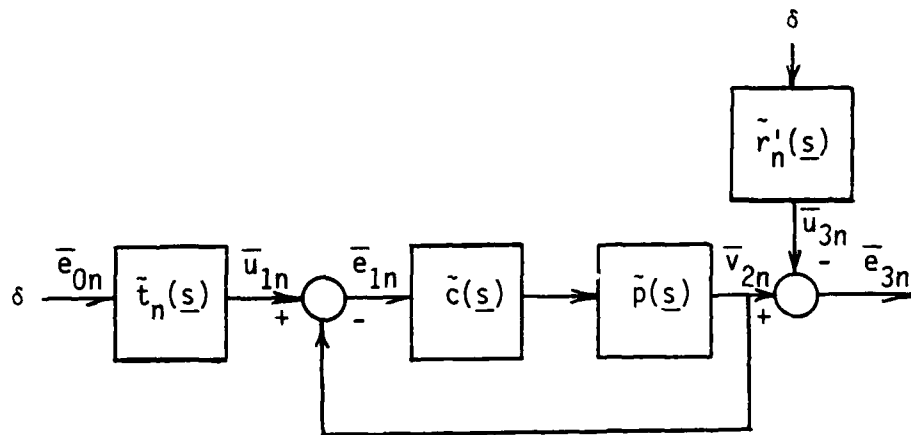
b. Intermediate Transformation
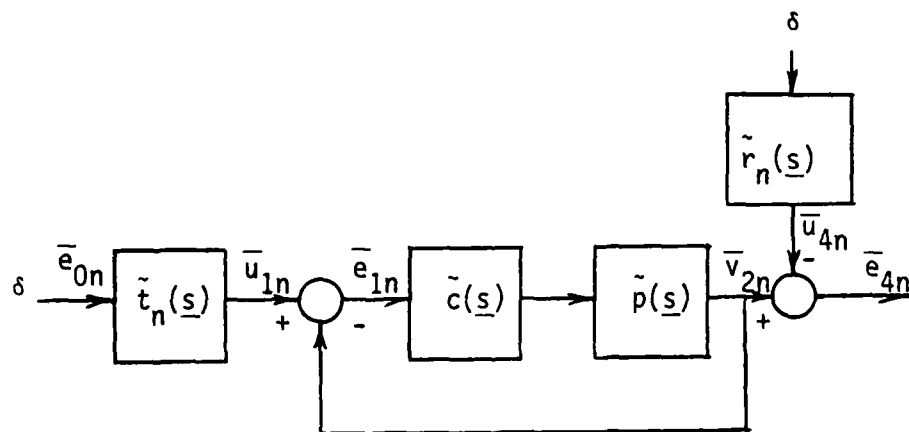
c. Final Transformation in New Domain

Figure 3. System Transformation

<u>Lemma 2.2</u>

Given the system in Figure 3, part a, and the functions described in Equations 2.26 and 2.27, the system in Figure 3, part b, will have an ouput $\bar{e}_3(\underline{t})$ with the property

$$\lim_{\underline{t}\to\infty} \{\bar{e}_3(t)\} = \lim_{t\to 0^+} \{v_2^n(t)\} \qquad <2.28>$$

<u>Proof</u>

The transfer function of the system in Figure 3, part a can be calculated to be

$$H_{v_2 e_0}(s) = t(s)\frac{c(s)p(s)}{1+c(s)p(s)} \qquad <2.29>$$

Then, invoking Lemma 2.1 would require a system with the transfer function

$$\overline{H}_n(\underline{s}) = \frac{1}{\underline{s}^{n+2}} \, t(\frac{1}{\underline{s}}) \, \frac{c(\frac{1}{\underline{s}})p(\frac{1}{\underline{s}})}{1+c(\frac{1}{\underline{s}})p(\frac{1}{\underline{s}})} - \sum_{i=0}^{n-1} \frac{1}{\underline{s}^{n-i+1}} \, v_2^i(0^+)$$

$$= \tilde{t}_n(s) \, \frac{\tilde{c}(\underline{s})\tilde{p}(\underline{s})}{1+\tilde{c}(\underline{s})\tilde{p}(\underline{s})} - \tilde{r}'_n(\underline{s}) \qquad <2.30>$$

via Equation 2.2, and invoking the multiplicative and additive properties, as well as Equations 2.26 and 2.27.

The system in Figure 3, part b, has the transfer function

$$\tilde{H}_{\bar{e}_{3n}\bar{e}_{0n}}(\underline{s}) = \tilde{t}_n(\underline{s}) \, \frac{\tilde{c}(\underline{s})\tilde{p}(\underline{s})}{1+\tilde{c}(\underline{s})\tilde{p}(\underline{s})} - \tilde{r}'(\underline{s}) \qquad <2.31>$$

Equations 2.30 and 2.31 are identical, revealing that Figure 3, part b, is in fact consistent with Lemma 2.1.

EOP

YBJ theory requires a system which, when stabilized, meets the required constraints. The property characterizing any stable system

is that asymptotically the response approaches zero. Considering this, if a different transfer function can be given that asymptotically equals the response of the system in Figure 3, part b, then the difference between these would be asymptotically zero. Therefore, a composite system equal to this difference would be stable.

A different transfer function with the asymptotic response $v_2^n(0^+)$ is the step function with weight $v_2^n(0^+)$,

$$\tilde{w}(\underline{s}) = \frac{v_2^n(0^+)}{\underline{s}} \qquad \qquad <2.32>$$

Then, the difference leading to stability would be the transfer function

$$\tilde{H}_{\overline{e}_{4n}\overline{e}_{0n}}(\underline{s}) = \tilde{H}_{\overline{e}_{3n}\overline{e}_{0n}}(\underline{s}) - \tilde{w}(\underline{s}) \qquad \qquad <2.33>$$

But, an equivalent model for this system would be to add $\tilde{w}(\underline{s})$ to $\tilde{r}_n'(\underline{s})$, creating a new function, $\tilde{r}_n(\underline{s})$, as follows.

$$\tilde{r}_n(\underline{s}) = \tilde{w}(\underline{s}) + r_n'(\underline{s})$$

$$= \frac{v_2^n(0^+)}{\underline{s}} + \sum_{i=0}^{n-1} \frac{1}{\underline{s}^{n-i+1}} v_2^i(0^+)$$

$$\tilde{r}_n(\underline{s}) = \sum_{i=0}^{n} \frac{1}{\underline{s}^{n-i+1}} v_2^i(0^+) \qquad \qquad <2.34>$$

This then leads to the following lemma.

Lemma 2.3

For the system in Figure 3, part a, let $t(s)$, $p(s)$, and the initial response constraints $\{y_0, y_1, \ldots, y_n, \ldots, y_j\}$ (y_i is finite, real) be given. Then, substitute the initial response constraints into $\tilde{r}_n(\underline{s})$, so that

$$\tilde{r}_n(\underline{s}) = \sum_{i=0}^{n} \frac{1}{s^{n-i+1}} y_i \qquad <2.35>$$

Then, the only compensators that will meet the constraint $v_2^n(0^+) = y_n$ which may possibly also meet the constraints $\{v_2^0(0^+), \ldots, v_2^{n-1}(0^+)\}$ = $\{y_0, \ldots, y_{n-1}\}$, are those which stabilize $\tilde{H}_{\bar{e}_{4n}\bar{e}_{0n}}(s)$ in Figure 3, part c, where $\tilde{r}_n(\underline{s})$ is written per Equation 2.35.

Proof

Transform the system in Figure 3, part a, to the one in Figure 3, part b. Then, in accordance with Lemma 2.2

$$\lim_{\underline{t}\to\infty} \{\bar{e}_3(\underline{t})\} = \lim_{t\to 0^+} \{v_2^n(t)\} = v_2^n(0^+) \qquad <2.36>$$

Also, let

$$\tilde{w}(\underline{s}) = \frac{y_n}{s} \qquad <2.37>$$

$$\lim_{\underline{t}\to\infty} \{\tilde{w}(\underline{t})\} = y_n$$

and define $\tilde{H}_{\bar{e}_{4n}\bar{e}_{0n}}(s)$ by Equation 2.33. Then, $\tilde{H}_{\bar{e}_{4n}\bar{e}_{0n}}(s)$ will be stabilized only if $\tilde{H}_{\bar{e}_{3n}\bar{e}_{0n}}(s)$ {which is equal to $\bar{e}_3(\underline{s})$} and $\tilde{w}(\underline{s})$ are asyptotically equal. This of course requires that

$$y_n = v_2^n(0^+) \qquad <2.38>$$

by equating Equations 2.36 and 2.37.

Therefore, consider a compensator $\tilde{c}(\underline{s})$ which leads to the system in Figure 3, part a, with inital response values $\{v_2^0(0^+), \ldots, v_2^{n-1}(0^+)\}$. Then, use these values, and let $v_2^n(0^+) = y_n$ for Equation 2.34. As such, if $\tilde{H}_{\bar{e}_{4n}\bar{e}_{0n}}(s)$ in Figure 3, part c, is stable, then Equation 2.38 applies.

Note that the above argument does not consider the constraints

$\{y_0, \ldots, y_{n-1}\}$. Thus, such a compensator may or may not meet these

constraints. However, if it does, then obviously these constraints

can be substituted for $\{v_2^0(0^+), \ldots, v_2^{n-1}(0^+)\}$ in Equation 2.34. Thus,

making this substitution, and invoking Equation 2.38 into Equation 2.34,

leads to Equation 2.35. Stabilizing the system in Figure 3, part c,

using Equation 2.35 therefore leads to a compensator that meets the

constraint y_n, and may (no guarantee yet!) meet $\{y_0, \ldots, y_{n-1}\}$. Also,

if it does cause the system to meet $\{y_0, \ldots, y_{n-1}\}$, then $\tilde{H}_{\overline{e}_{4n}\overline{e}_{0n}}(\underline{s})$ must

be stable when defining $\tilde{r}_n(\underline{s})$ according to Equation 2.35.
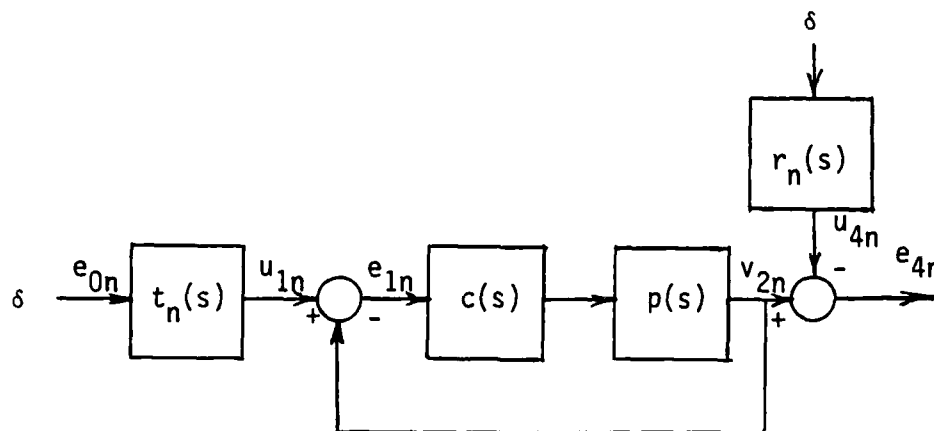
EOP

Figure 4. Final System Transformation

For the system in Figure 4, define

$$t_n(s) = \tilde{T}^{-1}\{\tilde{t}_n(\underline{s})\} = s^{n+2}t(s) \qquad <2.39>$$

$$r_n(s) = \tilde{T}^{-1}\{\tilde{r}_n(\underline{s})\} = \sum_{i=0}^{n} s^{n-i+1} y_i$$

per the transformation in Equation 2.17, using the inverse transform

and multiplicative properties, and per Equations 2.26 and 2.35. Then the following theorom applies.

Theorem 2.1

For the system in Figure 3, part a, let $t(s)$, $p(s)$, and the initial response constraints $\{y_0, y_1, \ldots y_n, \ldots, y_j\}$ be given, and define $t_n(s)$ and $r_n(s)$ per Equation 2.39. Then, the only compensators that will meet the constraints $v_2^n(0^+) = y_n$, which may possibly also meet the constraints $\{v_2^0(0^+), v_2^1(0^+), \ldots, v_2^{n-1}(0^+)\} = \{y_0, y_1, \ldots y_{n-1}\}$, are those which stabilize $H_{e_{4n}e_{0n}}(s)$ in Figure 4.

Proof

The additive and multiplicative properties dictate that transforming a system from the s-domain to the $\underline{s}$-domain, and visa versa, may be accomplished by transforming the individual transfer functions within the system separately. Therefore, the system in Figure 4 is the s-domain equivalent to the $\underline{s}$-domain system in Figure 3, part c. Also, by the stability transformation property, these systems are either both stable, or both not stable. Therefore, since the statement in Lemma 2.3 relies only on the stability of the $\underline{s}$-domain system, an equivalent statement can be made concerning the system in Figure 4.

EOP

With Theorem 2.1 in hand, the system in Figure 4 can be used for the application of the YBJ theory to find the stabilizing compensators that meet the initial response constraints. This will be accomplished in Chapter III.

2.5 Restrictions Due to IVT and FVT

The system in Figure 4 is the model from which a complete stabilizing parameterization is to be derived in the next chapter. However, this model is based upon the mathematical transformation in Section 2.2, which incorporates the IVT and FVT. At that time it was mentioned that both of these theorems have restrictions on their application. The intention here is to prove that this model will always be within the bounds of these restrictions, so that a complete parameterization can in fact be derived based upon it.

First consider the IVT, which states that if $H(s)$ is strictly proper, then

$$\lim_{t \to 0^+} \{h(t)\} = \lim_{s \to \infty} \{sH(s)\} \qquad \text{<2.40>}$$

In effect, this (strictly proper) says that there must be a finite initial value in order to apply the theorom. If $H(s)$ is not strictly proper, the initial value is undefined (infinite), and therefore equality cannot hold, even though the IVT will also yield an infinite initial value. Thus, this restriction simply requires that the constraints $\{y_0, y_1, \ldots, y_j\}$ be finite.

The FVT states that if $\overline{H}_n(\underline{s})$ has poles that lie entirely within the left half plane, except possibly for a simple pole at $s=0$ (see reference (8), p. 714), then

$$\lim_{\underline{t} \to \infty} \{\overline{h}_n(\underline{t})\} = \lim_{\underline{s} \to 0} \{\underline{s}\overline{H}_n(\underline{s})\} \qquad \text{<2.41>}$$

For the system model here,

$$\bar{H}_n(\underline{s}) = \tilde{H}_{\bar{v}_{2n}\bar{e}_{0n}}(\underline{s}) - \tilde{r}'_n(\underline{s})$$

$$= \tilde{H}_{\bar{v}_{2n}\bar{e}_{0n}}(\underline{s}) - \bar{r}_n(\underline{s}) - \frac{y_n}{\underline{s}}$$

$$= \tilde{H}_{\bar{e}_{4n}\bar{e}_{0n}}(s) - \frac{y_n}{\underline{s}} \tag{2.42}$$

and $\tilde{H}_{\bar{e}_{4n}\bar{e}_{0n}}(\underline{s})$ will always be stabilized. Thus, since $\bar{H}_n(\underline{s})$ is the

sum of proper functions, it will be proper. Also, its poles will be

those of its components, unless some are cancelled. Therefore, it will

be stable, except for possibly a simple pole at s=0, so the FVT

restrictions will always be met also.

2.6 Summary

Stabilizing the system model in Figure 4 has been shown to be

N & S to guarantee that the n-th derivative initial condition be met,

where the 0-th through (n-1)th initial conditions could possibly also

simultaneously be met. It will be shown in the next chapter that

meeting minor restrictions will guarantee the first n constraints are

met simultaneously when meeting the n-th is accomplished.

Another feature of this model is that the feedback loop portion

of the system is identical to that in the original system. This is

important since it is also desired to stabilize this loop, and convenient

because previous work in YBJ (10) has already solved this problem.

CHAPTER III

INITIAL RESPONSE THEOREM AND PARAMETERIZATION

3.1 Introduction

The IRT and IRP are derived in this chapter. The proceeding

derivations are based on the system in Figure 4 of Chapter II.

3.2 Definitions and Properties

Some general definitions and properties from reference (10) are

needed for the work in this chapter.

Definition 3.1

Given two transfer functions, $x(s)$ and $y(s)$, then $y(s)$ divides $x(s)$ if

$$\frac{x(s)}{y(s)} = y'(s) \qquad\qquad <3.1>$$

where $y'(s)$ is stable. Notationally, this is written $y|x$.

Definition 3.2

Two stable transfer functions $x(s)$ and $y(s)$ are RHP coprime, or simply

coprime, if they have no common RHP zeros. Notationally, this is written

$(x,y) = 1$.

Definition 3.3

The transfer functions $x(s)$ is said to be miniphase if it is stable,

and has a stable inverse.

Property 3.1

If $x(s)$ and $y(s)$ are stable and coprime, then there exist stable $u(s)$

and $v(s)$ such that

$$u(s)x(s) + v(s)y(s) = 1 \qquad\qquad <3.2>$$

Also, if <3.2> is valid for stable $x(s)$ and $y(s)$, then they are coprime.

Property 3.2

Let

$$r(s) = \frac{x_r(s)}{y_r(s)} \tag{3.3}$$

where $x_r(s)$ and $y_r(s)$ are both stable, but may not be coprime. Also, let

$$r(s) = \frac{n_r(s)}{d_r(s)} \tag{3.4}$$

be a coprime stable fractional representation for the same $r(s)$. Then there exists a stable $k(s)$ such that

$$x_r(s) = n_r(s)k(s)$$

and

$$y_r(s) = d_r(s)k(s) \tag{3.5}$$

Theorom 3.1 - Stabilization Theorem

For the feedback loop in Figure 2 with transfer function $H_{v_2 u_1}(s)$, and therefore also for the equivalent feedback loop in Figure 4 with transfer function $H_{v_{2n} u_{1n}}(s)$, let the plant have a coprime fractional representation

$$p(s) = \frac{n_p(s)}{d_p(s)} \tag{3.6}$$

There then exists stable $u_p(s)$ and $v_p(s)$ such that

$$u_p(s)n_p(s) + v_p(s)d_p(s) = 1 \tag{3.7}$$

Then for any stable $w(s)$ such that $w(s)n_p(s) + v_p(s)$ is not identically zero, the compensator

$$c(s) = \frac{\{-w(s)d_p(s) + u_p(s)\}}{\{w(s)n_p(s) + v_p(s)\}} = \frac{n_c(s)}{d_c(s)} \tag{3.8}$$

stabilizes the feedback loop and yields a coprime fractional

representation on

$$p(s)c(s) = \frac{\{n_p(s)n_c(s)\}}{\{d_p(s)d_c(s)\}} \qquad <3.9>$$

Conversely, every such stabilizing compensator is of this form for some stable $w(s)$. Note that Equation 3.9 guarantees that there are no pole-zero cancellations between the plant and the compensator.

Property 3.3

For the same feedback loop, if $c(s)$ is described by Equation 3.8 then

$$H_{v_2 u_1}(s) = H_{v_{2n} u_{1n}}(s) = -w(s)n_p(s)d_p(s) + u_p(s)n_p(s) \qquad <3.10>$$

3.3 Problem Formulation

Both the feedback loop and the entire system in Figure 4 must be stabilized. Theorom 3.1 describes the parameterization stabilizing the feedback loop portion, so the desired compensator stabilizing both quantities must be a subset of this form. Referring to Figure 4, define

$$z_n(s) = e_{4n}(s) = \frac{n_{z_n}(s)}{d_{z_n}(s)} \qquad <3.11>$$

and invoking Equation 3.10

$$z_n(s) = t_n(s) \{-w_n(s)n_p(s)d_p(s) + u_p(s)n_p(s)\} - r_n(s) \qquad <3.12>$$

where $t_n(s)$ and $r_n(s)$ are described per Equation 2.30. Thus, the objective here is to stabilize $z_n(s)$ whenever $w_n(s)$ is stable. A parameterization that stabilizes $z_n(s)$ guarantees that $v_2^n(0^+) = y_n$. It may not necessarily guarantee that any other derivative constraints are met, however. This problem will be considered later in this chapter.

3.4 System Properties and Definitions

In this section groundwork is laid in preparation for analysis
of the system in Figure 4 and Equation 3.12, leading up to the IRT
and IRP. Several properties and definitions will be required.

From Equation 3.12, define

$$t_n(s) = \frac{n_{t_n}(s)}{d_{t_n}(s)} \qquad\qquad <3.13>$$

and

$$r_n(s) = \frac{n_{r_n}(s)}{d_{r_n}(s)}$$

to be stable coprime fractional representations of $t_n(s)$ and $r_n(s)$.
As such, by Property 3.1 there exist stable u's and v's such that

$$u_{t_n}(s)n_{t_n}(s) + v_{t_n}(s)d_{t_n}(s) = 1 \qquad\qquad <3.14>$$

and

$$u_{r_n}(s)n_{r_n}(s) + v_{r_n}(s)d_{r_n}(s) = 1$$

Then, based on the assumption that the plant is a proper rational
function, the following property holds.

Property 3.4

The plant and $r_n(s)$ are such that

$$(d_{r_n}(s), d_p(s)) = 1 \qquad\qquad <3.15>$$

and therefore there exist stable $u_{r_np}(s)$ and $v_{r_np}(s)$ such that

$$u_{r_np}(s)d_{r_n}(s) + v_{r_np}(s)d_p(s) = 1 \qquad\qquad <3.16>$$

Proof

The plant is assumed to be proper, so that $r\{p(s)\} \geq 0$. Thus, the hurwitz polynomial which would be used as the common divisor of both the numerator and denominator to create $n_p(s)$ and $d_p(s)$ must be of the same degree as the denominator. Therefore, $r\{d_p(s)\} = 0$. Such a function could have zeros only at finite values of s.

On the other hand, $r_n(s)$ was described in Equation 2.39 to be a polynomial in s, such that, considering that some y_i's could be equal to zero

$$-n \leq r\{r_n(s)\} \leq 0 \qquad <3.17>$$

If $r\{r_n(s)\} = 0$, then $r_n(s)$ is identically zero, so $n_{r_n}(s)$ must equal zero. Therefore,

$$v_{r_n}(s)d_{r_n}(s) = 1 \qquad <3.18>$$

in which case $d_{r_n}(s)$ is proven coprime with $d_p(s)$ by letting $u_{r_n p}(s)$ equal zero and $v_{r_n p}(s)$ equal $v_{r_n}(s)$. Also, if $r\{r_n(s)\} < 0$, then obviously $r\{d_{r_n}(s)\} > 0$, and $d_{r_n}(s)$ has a constant as it numerator. In this case, $d_{r_n}(s)$ must be zero only at infinite values of s. Thus, $d_{r_n}(s)$ and $d_p(s)$ never have common RHP zeros, and therefore are coprime.

EOP

Definition 3.4

Define $a_n(s)$ such that

$$a_n(s) = \frac{n_p(s)}{d_{t_n}(s)} = \frac{n_{a_n}(s)}{d_{a_n}(s)} \quad ; \quad u_{a_n}(s)n_{a_n}(s) + v_{a_n}(s)d_{a_n}(a) = 1 \qquad <3.19>$$

This function has the same finite zeros as the plant, and the same finite poles as $t_n(s)$, less those that cancel.

Definition 3.5

Define $b_n(s)$ such that

$$b_n(s) = \frac{d_p(s)}{d_{t_n}(s)} = \frac{n_{b_n}(s)}{d_{b_n}(s)} \quad ; \quad u_{b_n}(s)n_{b_n}(s) + v_{b_n}(s)d_{b_n}(s) = 1 \qquad <3.20>$$

This function has the poles of $p(s)$ as its finite zeros, and the same finite poles as $t_n(s)$, less those that cancel.

The above definitions will be needed later in this chapter, and lead to the following two properties.

Property 3.5

Invoking Property 3.2 into Equations 3.19 and 3.20 leads to the conclusion that there exist stable $m_{a_n}(s)$ and $m_{b_n}(s)$ which satisfy the following.

$$m_{a_n}(s)n_{a_n}(s) = n_p(s) \quad ; \quad m_{a_n}(s)d_{a_n}(s) = d_{t_n}(s)$$

$$m_{b_n}(s)n_{b_n}(s) = d_p(s) \quad ; \quad m_{b_n}(s)d_{b_n}(s) = d_{t_n}(s) \qquad <3.21>$$

Property 3.6

The functions $m_{a_n}(s)$ and $m_{b_n}(s)$ are coprime.

Proof

Per Equation 3.7

$$u_p(s)n_p(s) + v_p(s)d_p(s) = 1$$

Then, substituting for $n_p(s)$ and $d_p(s)$ from Equation 3.21 yields

$$u_p(s)m_{a_n}(s)n_{a_n}(s) + v_p(s)m_{b_n}(s)n_{b_n}(s) = 1$$

$$\{u_p(s)n_{a_n}(s)\}m_{a_n}(s) + \{v_p(s)n_{b_n}(s)\}m_{b_n}(s) = 1 \tag{3.22}$$

which proves $m_{a_n}(s)$ and $m_{b_n}(s)$ are coprime by Property 3.1.

EOP

Definition 3.6

Define $d_{c_n}(s)$ as follows.

$$d_{c_n}(s) = \frac{d_{b_n}(s)}{m_{a_n}(s)} \tag{3.23}$$

Property 3.7

The function $d_{c_n}(s)$ is stable.

Proof

By definition

$$d_{c_n}(s) = \frac{d_{b_n}(s)}{m_{a_n}(s)}$$

Multiplying by one, in the form of Equation 3.22 leads to

$$d_{c_n}(s) = \frac{d_{b_n}(s)}{m_{a_n}(s)} \{u_p(s)n_{a_n}(s)m_{a_n}(s) + v_p(s)n_{b_n}(s)m_{b_n}(s)\}$$

$$= d_{b_n}(s)u_p(s)n_{a_n}(s) + \frac{d_{b_n}(s)v_p(s)n_{b_n}(s)m_{b_n}(s)}{m_{a_n}(s)} \tag{3.24}$$

But,

$$\frac{m_{b_n}(s)d_{b_n}(s)}{m_{a_n}(s)d_{a_n}(s)} = \frac{d_{t_n}(s)}{d_{t_n}(s)}$$

so

$$\frac{m_{b_n}(s)}{m_{a_n}(s)} = \frac{d_{a_n}(s)}{d_{b_n}(s)} \qquad\qquad <3.25>$$

Then, substituting Equation 3.25 into Equation 3.24 yield

$$d_{c_n}(s) = d_{b_n}(s)u_p(s)n_{a_n}(s) + \frac{d_{b_n}(s)v_p(s)n_{b_n}(s)d_{a_n}(s)}{d_{b_n}(s)}$$

$$d_{c_n}(s) = d_{b_n}(s)u_p(s)n_{a_n}(s) + v_p(s)n_{b_n}(s)d_{a_n}(s)$$

which is stable since it is the product and sum of stable functions.
EOP

Note that qualitatively $d_{c_n}(s)$ can be thought of having as its finite zeros the finite poles of $t_n(s)$ that are not common with the finite poles and zeros of the plant. Two more properties will be needed concerning $d_{c_n}(s)$ before the preceeding definitions and properties are used to analyze Equation 3.12.

Property 3.8

$$d_{t_n}(s) = m_{a_n}(s)m_{b_n}(s)d_{c_n}(s) \qquad\qquad <3.26>$$

Proof

From Equation 3.21

$$m_{b_n}(s)d_{b_n}(s) = d_{t_n}(s) \tag{3.27}$$

and from Definition 3.6

$$d_{c_n}(s) = \frac{d_{b_n}(s)}{m_{a_n}(s)}$$

or

$$d_{c_n}(s)m_{a_n}(s) = d_{b_n}(s) \tag{3.28}$$

Therefore, substituting $d_{b_n}(s)$ in Equation 3.28 into Equation 3.27 yields

$$m_{a_n}(s)m_{b_n}(s)d_{c_n}(s) = d_{t_n}(s)$$

EOP

Property 3.9

The function $d_{c_n}(s)$ is coprime with the quantity $n_{a_n}(s)n_{b_n}(s)n_{t_b}(s)$, so that there exists stable $u_{s_n}(s)$ and $v_{s_n}(s)$ where

$$u_{s_n}(s)n_{a_n}(s)n_{b_n}(s)n_{t_n}(s) + v_{s_n}(s)d_{c_n}(s) = 1 \tag{3.29}$$

Proof

This proof is in four stages. First, $(d_{c_n}(s), n_{t_n}(s)) = 1$ is proved as follows.

$$u_{t_n}(s)n_{t_n}(s) + v_{t_n}(s)d_{t_n}(s) = 1$$

and invoking Property 3.8 leads to

$$\{u_{t_n}(s)\}n_{t_n}(s) + \{v_{t_n}(s)m_{a_n}(s)m_{b_n}(s)\}d_{c_n}(s) = 1 \qquad <3.30>$$

Next, $(d_{c_n}(s), n_{b_n}(s)) = 1$ is shown as follows.

$$u_{b_n}(s)n_{b_n}(s) + v_{b_n}(s)d_{b_n}(s) = 1$$

and substituting for $d_{b_n}(s)$ from Equation 3.21 yields

$$u_{b_n}(s)n_{b_n}(s) + v_{b_n}(s)\frac{d_{t_n}(s)}{m_{b_n}(s)} = 1$$

Then, substituting for $d_{t_n}(s)$ according to Equation 3.26 leads to

$$\{u_{b_n}(s)\}n_{b_n}(s) + \{v_{b_n}(s)m_{a_n}(s)\}d_{c_n}(s) = 1 \qquad <3.31>$$

Thirdly, $(d_{c_n}(s), n_{a_n}(s)) = 1$ can be shown as follows.

$$u_{a_n}(s)n_{a_n}(s) + V_{a_n}(s)d_{a_n}(s) = 1$$

and substituting for $d_{a_n}(s)$ according to Equation 3.21 and invoking

Equation 3.26 leads to

$$u_{a_n}(s)n_{a_n}(s) + v_{a_n}(s)\frac{d_{t_n}(s)}{m_{a_n}(s)} = 1$$

$$\{u_{a_n}(s)\}n_{a_n}(s) + \{v_{a_n}(s)m_{b_n}(s)\}d_{c_n}(s) = 1 \qquad <3.32>$$

Finally, multiplying the three preceeding results together leads to
the following result. Note that to save space, the variable s and
subscript n have been omitted from the equations.

$$1 = 1 \times 1 \times 1$$

$$= \{u_{t_n} n_{t_n} + v_{t_n} m_{a_n} m_{b_n} d_{c_n}\}\{u_{b_n} n_{b_n} + v_{b_n} m_{a_n} d_{c_n}\}\{u_{a_n} n_{a_n} + v_{a_n} m_{b_n} d_{c_n}\}$$

$$= \{u_{t_n} n_{t_n} u_{b_n} n_{b_n} + u_{t_n} n_{t_n} v_{b_n} m_{a_n} d_{c_n} + v_{t_n} m_{a_n} m_{b_n} d_{c_n} u_{b_n} n_{b_n}$$

$$+ v_{t_n} m_{a_n} m_{b_n} d_{c_n} v_{b_n} m_{a_n} d_{c_n}\}\{u_{a_n} n_{a_n} + v_{a_n} m_{b_n} d_{c_n}\}$$

$$1 = \{u_{t_n} u_{b_n} u_{a_n}\} n_{a_n} n_{b_n} n_{t_n} + \{u_{t_n} n_{t_n} u_{b_n} n_{b_n} v_{a_n} m_{b_n}$$

$$+ u_{t_n} n_{t_n} v_{b_n} m_{a_n} u_{a_n} n_{a_n} + u_{t_n} n_{t_n} v_{b_n} m_{a_n} d_{c_n} v_{a_n} m_{b_n}$$

$$+ v_{t_n} m_{a_n} m_{b_n} u_{b_n} n_{b_n} u_{a_n} n_{a_n} + v_{t_n} m_{a_n} m_{b_n} d_{c_n} u_{b_n} n_{b_n} v_{a_n} m_{b_n}$$

$$+ v_{t_n} m_{a_n} m_{b_n} v_{b_n} m_{a_n} d_{c_n} u_{a_n} n_{a_n} + v_{t_n} m_{a_n} m_{b_n} d_{c_n} v_{b_n} m_{a_n} d_{c_n} v_{a_n} m_{b_n}\} d_{c_n}$$

The quantities inside the brackets are stable since they are the products and sums of stable functions. Therefore $d_{c_n}(s)$ is coprime with the quantity $n_{a_n}(s) n_{b_n}(s) n_{t_n}(s)$.

EOP

3.5 System Analysis

Using the preceeding definitions and properties, Equation 3.12 will now be analyzed to find the N & S conditions that guarantee stability of both $z_n(s)$ and $w_n(s)$ at the same time. The following property is presented as an intermediate step towards more specific criteria for stability.

Property 3.10

Equation 3.12 admits stable $z_n(s)$ and $w_n(s)$ if and only if

$$g_n(s) = \frac{u_p(s) n_p(s) n_{t_n}(s) - x_n(s)}{m_{a_n}(s)} \qquad <3.33>$$

and

$$k_n(s) = \frac{u_p(s)n_p(s)n_{t_n}(s) - x_n(s)}{m_{b_n}(s)}$$

<3.34>

are stable, where

$$x_n(s) = \frac{n_{r_n}(s)d_{t_n}(s)}{d_{r_n}(s)}$$

<3.35>

and where $x_n(s)$ is stable.

<u>Proof</u>

Equation 3.12 is rearranged as follows.

$$z_n(s) = t_n(s)\{-w_n(s)n_p(s)d_p(s) + u_p(s)n_p(s)\} - r_n(s)$$

$$= \frac{n_{t_n}(s)}{d_{t_n}(s)} \{-w_n(s)n_p(s)d_p(s) + u_p(s)n_p(s)\} - \frac{n_{r_n}(s)}{d_{r_n}(s)}$$

$$z_n(s)d_{t_n}(s)d_{r_n}(s) = n_{t_n}(s)d_{r_n}(s)\{-w_n(s)n_p(s)d_p(s) + u_p(s)n_p(s)\}$$

$$- n_{r_n}(s)d_{t_n}(s)$$

and

$$z_n(s)d_{t_n}(s)d_{r_n}(s) + w_n(s)n_p(s)d_p(s)n_{t_n}(s)d_{r_n}(s) - u_p(s)n_p(s)n_{t_n}(s)d_{r_n}(s)$$

$$= -n_{r_n}(s)d_{t_n}(s)$$

$$z_n(s)d_{t_n}(s) + w_n(s)n_p(s)d_p(s)n_{t_n}(s) - u_p(s)n_p(s)n_{t_n}(s) = \frac{-n_{r_n}(s)d_{t_n}(s)}{d_{r_n}(s)}$$

$$= -x_n(s)$$

$x_n(s)$ must be stable since it is equal to the product and sum of stable functions when $z_n(s)$ and $w_n(s)$ are stable. Then, substituting Equations 3.26 and 3.21 into the above equation and rearranging leads to (omitting the variables and subscript n)

$$zm_a m_b d_c + wm_a n_a m_b n_b n_t - u_p n_p n_t = -x$$

$$(zd_c + wn_a n_b n_t)m_a m_b = u_p n_p n_t - x$$

$$zd_c + wn_a n_b n_t = \frac{u_p n_p n_t - x}{m_a m_b} \qquad <3.36>$$

If $z_n(s)$ and $w_n(s)$ are stable, then the left side, and therefore the right side of Equation 3.36 is stable. Also, in this case, if Equation 3.36 is multiplied by either $m_{a_n}(s)$ or $m_{b_n}(s)$

$$(zd_c + wn_a n_b n_t)m_a = \frac{u_p n_p n_t - x}{m_b} = k \qquad <3.37>$$

$$(zd_c + wn_a n_b n_t)m_b = \frac{u_p n_p n_t - x}{m_a} = g \qquad <3.38>$$

the left side of the equation is still stable. Therefore $k_n(s)$ and $g_n(s)$ are stable. Thus far we have proven that in order for a solution to exist, the right side of Equations 3.36, 3.37, and 3.38 must be stable. To prove that stable $k_n(s)$ and $g_n(s)$ is sufficient for the existence of stable $z_n(s)$ and $w(s)$, first multiply the right side of Equation 3.36 by one, in the form of Equation 3.29.

$$zd_c + wn_a n_b n_t = \frac{u_p n_p n_t - x}{m_a m_b} \{u_s n_a n_b n_t + v_s d_c\}$$

$$= \{\frac{(u_p n_p n_t - x)v_s}{m_a m_b}\}dc + \{\frac{(u_p n_p n_t - x)u_s}{m_a m_b}\}n_a n_b n_t$$

Therefore, if Equation 3.36 is stable, then particular stable solutions obviously exist for $z_n(s)$ and $w_n(s)$. Also, <3.36> is stable whenever $g_n(s)$ and $k_n(s)$ are stable. This can be proven by multiplying by Equation 3.22, as follows.

$$\frac{u_p n_p n_t - x}{m_a m_b} = \frac{u_p n_p n_t - x}{m_a m_b} \{u_p n_a m_a + v_p n_b m_b\}$$

$$= \frac{u_p n_p n_t - x}{m_b} u_p n_a + \frac{u_p n_p n_t - x}{m_a} v_p n_b$$

$$= k u_p n_a + g v_p n_b \qquad \qquad <3.39>$$

Thus, if $k_n(s)$ and $g_n(s)$ are stable, the right side of Equation 3.29 is stable, and therefore, also the left. Finally, to show that $k_n(s)$ and $g_n(s)$ must be stable if 3.36 is stable, let

$$\frac{u_p(s)n_p(s)n_{t_n}(s) - x_n(s)}{m_{a_n}(s)m_{b_n}(s)} = z_n'(s) \qquad \qquad <3.40>$$

where $z_n'(s)$ stable. Then multiplying by $m_{a_n}(s)$ yields

$$k_n(s) = m_{a_n}(s)z_n'(s) \qquad \qquad <3.41>$$

which must be stable if $z_n'(s)$ is stable. Also, multiplying <3.40> by $m_{b_n}(s)$ leads to

$$g_n(s) = m_{b_n}(s)z_n'(s) \qquad \qquad <3.42>$$

which is stable. Therefore, it is both necessary and sufficient that $x_n(s)$, $k_n(s)$ and $g_n(s)$ are stable in order for $z_n(s)$ and $w_n(s)$ to both

be stable.

EOP

The above property defines the basic mathematical commodities that must be stable in order for a solution to Equation 3.12 to exist. However, the commodities are derived mathematically from the original system, and give a designer no "feel" as to their meaning. In practice, one would like the necessary and sufficient conditions to be related simply to basic system properties which are not derived mathematically. With this goal in mind, the quantities given in Property 3.10 are further analyzed as follows.

Property 3.11

$$\frac{x_n(s)}{m_{b_n}(s)}$$

is stable.

Proof

Again, omitting the variable s and subscript n for compactness

$$x = \frac{n_r d_t}{d_r}$$

by definition, so that

$$\frac{x}{m_b} = \frac{n_r d_t}{d_r m_b}$$

Then, using Property 3.8 yields

$$\frac{x}{m_b} = \frac{n_r m_a m_b d_c}{d_r m_b} = \frac{n_r m_a d_c}{d_r}$$

Multiplying by one in the form of Equation 3.16 leads to

$$\frac{x}{m_b} = \frac{n_r m_a d_c}{d_r} \{u_{rp} d_r + v_{rp} d_p\}$$

$$= n_r m_a d_c u_{rp} + \frac{n_r m_a d_c v_{rp} d_p}{d_r}$$

Then, substituting for $d_p(s)$ according to Equation 3.21 leads to

$$\frac{x}{m_b} = n_r m_a d_c u_{rp} + \frac{n_r m_a d_c v_{rp} m_b n_b}{d_r}$$

$$= n_r m_a d_c u_{rp} + \frac{n_r m_a m_b d_c}{d_r} v_{rp} n_b$$

$$\frac{x}{m_b} = n_r m_a d_c u_{rp} + x v_{rp} n_b \tag{3.43}$$

The right side of Equation 3.43 is stable, and therefore so is the left. EOP

Property 3.11 in itself provides no further immediate insights, but it does lead directly to the following lemma which does.

Lemma 3.1

Equation 3.34 admits stable solutions iff

$$(d_{t_n}(s), d_p(s)) = 1 \tag{3.44}$$

If this is the case, then there exist stable $u_{t_n p}(s)$ and $v_{t_n p}(s)$ such that

$$u_{t_n p}(s) d_{t_n}(s) + v_{t_n p}(s) d_p(s) = 1 \tag{3.45}$$

Proof

Rearranging Equation 3.34 as follows,

$$k_n(s) = \frac{u_p(s)n_p(s)n_{t_n}(s) - x_n(s)}{m_{b_n}(s)}$$

$$= \frac{u_p(s)n_p(s)n_{t_n}(s)}{m_{b_n}(s)} - \frac{x_n(s)}{m_{b_n}(s)} \qquad \langle 3.46 \rangle$$

$$k_n(s) + \frac{x_n(s)}{m_{b_n}(s)} = \frac{u_p(s)n_p(s)n_{t_n}(s)}{m_{b_n}(s)} \qquad \langle 3.47 \rangle$$

If $k_n(s)$ is to be stable, then the left side of $\langle 3.47 \rangle$ is stable by invoking Property 3.11. Therefore, the right side must be stable also. Stability of the right side of $\langle 3.47 \rangle$ sufficiently guarantees stability of $k_n(s)$, since in this case Equation 3.46 represents $k_n(s)$ as the sum of two stable functions.

Referring to Equation 3.44, if either $d_{t_n}(s)$ or $d_p(s)$ are miniphase, then they are obviously coprime. This can be proven by multiplying the miniphase function by its inverse (yielding 1) and the other function by zero. Thus, we have narrowed the proof down to the case where both functions have zeros in the RHP.

Equation 3.20 effectively cancels the common zeros between $d_p(s)$ and $d_{t_n}(s)$ in creating $n_{b_n}(s)$ and $d_{b_n}(s)$, which in effect are then equal to the first two functions less the common zeros. Then, by Equation 3.21,

$$m_{b_n}(s) = \frac{d_p(s)}{n_{b_n}(s)} = \frac{d_{t_n}(s)}{d_{b_n}(s)}$$

Thus, $m_{b_n}(s)$ is the original function divided by itself without the common zeros, yielding a function only containing all zeros that are common between $d_p(s)$ and $d_{t_n}(s)$.

If $d_{t_n}(s)$ and $d_p(s)$ are not coprime, then the above argument dictates that $m_{b_n}(s)$ has RHP zeros, and is not miniphase. Then, to stabilize <3.47>, it is necessary that $(m_{b_n} | u_p n_p n_{t_n})$. But, $d_p(s)$ is coprime with $u_p(s) n_p(s)$, because

$$\{1\} u_p(s) n_p(s) + \{v_p(s)\}\, d_p(s) = 1$$

Thus, $m_{b_n}(s)$ is also coprime since it only has zeros from $d_p(s)$. Therefore, $m_{b_n}(s)$ must divide $n_{t_n}(s)$, and in order for this to occur, it must have common zeros. But, $m_{b_n}(s)$ also derived its zeros strictly from $d_{t_n}(s)$ which has no common RHP zeros with $n_{t_n}(s)$. Therefore, $m_{b_n}(s)$ can not divide $n_{t_n}(s)$. This proves that $m_{b_n}(s)$ must be miniphase, also proving that $d_p(s)$ and $d_{t_n}(s)$ must be coprime.

EOP

Lemma 3.1 provides the necessary and sufficient condition in order to stabilize Equation 3.34. It is interesting to note that the lemma simply requires that the plant not have any common RHP poles with the input, less perhaps up to $(n + 2)$ poles at $s = 0$ of the input. For most systems, this of course should present no difficulty, and therefore is not a very strict condition. The condition necessary to

stabilize Equations 3.33 and 3.35 must still be derived. The following property leads towards Lemma 2, which does this.

Property 3.12

If $k_n(s)$ is stable, then $g_n(s)$ is stable iff $d_{r_n}(s)$ divides $d_{c_n}(s)$, in which case $x_n(s)$ is also stable.

Proof

If $d_{r_n}(s)$ is miniphase, it will always divide $d_{c_n}(s)$. Attention must then be focused on the case where it is not miniphase. Then, using Equation 3.21

$$\frac{u_p(s)n_p(s)n_{t_n}(s)}{m_{a_n}(s)} = \frac{u_p(s)m_{a_n}(s)n_{a_n}(s)n_{t_n}(s)}{m_{a_n}(s)}$$

$$= u_p(s)n_{a_n}(s)n_{t_n}(s)$$

which is stable. Thus, rearranging <3.33>,

$$g_n(s) = \frac{u_p(s)n_p(s)n_{t_n}(s) - x_n(s)}{m_{a_n}(s)}$$

$$= u_p(s)n_{a_n}(s)n_{t_n}(s) - \frac{x_n(s)}{m_{a_n}(s)} \qquad <3.48>$$

$$g_n(s) + u_p(s)n_{a_n}(s)n_{t_n}(s) = -\frac{x_n(s)}{m_{a_n}(s)} \qquad <3.49>$$

Equation 3.49 shows that the right side must be stable if $g_n(s)$, and therefore the left side, is stable. Equation 3.48 shows that if

$$\frac{x_n(s)}{m_{a_n}(s)}$$

is stable, then so is $g_n(s)$. Thus, stabilizing this term is necessary and sufficient to stabilize $g_n(s)$. Then, invoking Equations 3.35 and 3.26 leads to

$$\frac{x_n(s)}{m_{a_n}(s)} = \frac{n_{r_n}(s)d_{t_n}(s)}{d_{r_n}(s)m_{a_n}(s)}$$

$$= \frac{n_{r_n}(s)m_{a_n}(s)m_{b_n}(s)d_{c_n}(s)}{d_{r_n}(s)m_{a_n}(s)}$$

$$= \frac{n_{r_n}(s)m_{b_n}(s)d_{c_n}(s)}{d_{r_n}(s)} \qquad \text{<3.50>}$$

which must be stable. Analyzing <3.50>, one notes that $d_{r_n}(s)$ must divide the numerator, since we are concerned with $d_{r_n}(s)$ not miniphase. But, it can not divide $n_{r_n}(s)$ even partially since they are coprime. Also, in the case that $k_n(s)$ is stable, during the proof of Lemma 3.1, it was shown that $m_{b_n}(s)$ is miniphase, so $d_{r_n}(s)$ cannot divide $m_{b_n}(s)$ either. Therefore, in order for $g_n(s)$ to be stable when $k_n(s)$ is stable, $d_{r_n}(s)$ must divide $d_{c_n}(s)$. Note that this condition also guarantees $x_n(s)$ is stable, since

$$x_n(s) = n_{r_n}(s)m_{a_n}(s)m_{b_n}(s) \frac{d_{c_n}(s)}{d_{r_n}(s)}$$

EOP

The above property still does not relate directly to system quantities, since $d_{c_n}(s)$ is derived. Thus, the following lemmas is given,

whose proof further analyzes the condition in Property 3.12 in order to determine what is physically meant by the statement.

Lemma 3.2

If $k_n(s)$ is stable, then $g_n(s)$ and $x_n(s)$ are both stable iff

$$r\{t(s)\} + r\{p(s)\} \leq M + 1 \qquad \text{<3.51>}$$

where M is the lowest order initial derivative constraint not equal to zero, i.e., $y_i = 0$ for $i < M$, and $y_M \neq 0$.

Proof

If $k_n(s)$ is stable, then by Property 3.12 $g_n(s)$ and $x_n(s)$ are stable iff $(d_{r_n} | d_{c_n})$. Thus, this condition must be analyzed to see when it occurs.

Before proceeding further however, the following shorthand notation is introduced which will prove useful. First, let $0_s(n)$ represent a stable (hurwitz) polynomial of order n, and let $0_u(n)$ represent a general polynomial of order n, where n is an integer. In this notation if n is less than zero, then

$$0(n) = \frac{1}{0(-n)} \qquad \text{<3.52>}$$

Equation 2.39 defined

$$r_n(s) = \sum_{i=0}^{n} s^{n-i+1} y_i$$

and $t_n(s) = s^{n+2} t(s)$

Let M be the lowest degree initial response constraint not equal to zero, so that y_M is not zero, but $\{y_0, \ldots, y_{M-1}\} = 0$.

Then,

$$r_n(s) = s^{n-M+1}y_M + \ldots + sy_n = 0_u(n-M+1) \qquad <3.53>$$

The coprime representation of $r_n(s)$ is then

$$r_n(s) = 0_u(n-M+1) = \frac{\dfrac{0_u(n-M+1)}{0_s(n-M+1)}}{\dfrac{1}{0_s(n-M+1)}} = \frac{n_{r_n}(s)}{d_{r_n}(s)}$$

and

$$d_{r_n}(s) = \frac{1}{0_s(n-M+1)} \qquad <3.54>$$

The plant is such that $r\{p(s)\} \geq 0$ by assumption. Therefore, the coprime representation of $p(s)$ is developed as follows.

$$p(s) = \frac{0_u(x_p)}{0_u(y_p)} \qquad x_p \leq y_p$$

$$p(s) = \frac{\dfrac{0_u(x_p)}{0_s(y_p)}}{\dfrac{0_u(y_p)}{0_s(y_p)}} = \frac{n_p(s)}{d_p(s)} \quad , \; x_p \leq y_p \qquad <3.55>$$

For the coprime representation of $t_n(s) = s^{n+2}t(s)$, it is necessary to know how many poles at $s = 0$ $t(s)$ has. Thus, let

$$t(s) = \frac{0_u(x_t)}{s^c \cdot 0_u(y_t)} \qquad <3.56>$$

where c is a non-negative integer, and where $0_u(y_t)$ represents a polynomial that is not divisable by s. Also, if c is positive, then $0_u(x_t)$ is not divisable by s. Then,

$$t_n(s) = \frac{s^{n+2}0_u(x_t)}{s^c 0_u(y_t)} = \frac{s^{n+2-c}0_u(x_t)}{0_u(y_t)}$$

There then are two possibilities for $d_{t_n}(s)$, as follows.

$$d_{t_n}(s) = \frac{0_u(y_t)}{0_s(\max\{y_t, n+2-c+x_t\})} \qquad <3.57>$$

when

$$n+2-c \geq 0$$

and

$$d_{t_n}(s) = \frac{s^{c-n-2} 0_u(y_t)}{0_s(\max\{x_t, c-n-2+y_t\})} \qquad <3.58>$$

when

$$c-n-2 > 0$$

The remainder of this proof must then be in two parts, since it is dependent on $d_{t_n}(s)$. First, consider when <3.57> applies. Then,

$$a_n(s) = \frac{n_p(s)}{d_{t_n}(s)} = n_p(s) \frac{1}{d_{t_n}(s)}$$

$$= \frac{0_u(x_p)}{0_s(y_p)} \cdot \frac{0_s(\max\{y_t, n+2-c+x_t\})}{0_u(y_t)} \qquad <3.59>$$

Coprime representations do not require cancellations between stable poles and zeros, but it does between unstable ones. Thus, let A be the number of common unstable poles and zeros in <3.59>, which must be contained within $0_u(x_p)$ and $0_u(y_t)$. Thus, let $0_u(x_p-A)$ and $0_u(y_t-A)$ represent the just mentioned polynomials without the A common terms, so that

$$a_n(s) = \frac{0_u(x_p-A)0_s(\max\{y_t, n+2-c+x_t\})}{0_u(y_t-A)0_s(y_p)}$$

The proper coprime representation of $n_{a_n}(s)$ is then

$$n_{a_n}(s) = \frac{0_u(x_p-A)0_s(\max\{y_t,n+2-c+x_t\})}{0_s(\max\{x_p-A+\max\{y_t,n+2-c+x_t\},y_t-A+y_p\})} \qquad <3.60>$$

To condense the equations, it will prove useful to let

$$\max\{y_t,n+2-c+x_t\} = B_1 \qquad <3.61>$$

and

$$\max\{x_p-A+\max\{y_t,n+2-c+x_t\},y_t-A+y_p\} = B_2$$

so that

$$n_{a_n}(s) = \frac{0_u(x_p-A)0_s(B_1)}{0_s(B_2)} \qquad <3.62>$$

and

$$d_{t_n}(s) = \frac{0_u(y_t)}{0_s(B_1)}$$

Then,

$$m_{a_n}(s) = \frac{n_p(s)}{n_{a_n}(s)} = \frac{0_u(x_p)}{0_s(y_p)} \cdot \frac{0_s(B_2)}{0_u(x_p-A)0_s(B_1)}$$

$$= \frac{0_u(A)0_s(B_2)}{0_s(y_p)0_s(B_1)} \qquad <3.63>$$

To calculate $d_{c_n}(s)$, Equation 3.26 is used, so that

$$d_{c_n}(s) = \frac{d_{t_n}(s)}{m_{a_n}(s)m_{b_n}(s)}$$

Earlier in this paper it was shown that $m_{b_n}(s)$ must be miniphase in order for a solution to exist. Thus, let $x_b = y_b$, so that

$$m_{b_n}(s) = \frac{0_s(x_b)}{0_s(y_b)} \qquad <3.64>$$

and

$$d_{c_n}(s) = \frac{0_u(y_t)}{0_s(B_1)} \cdot \frac{0_s(y_p)0_s(B_1)}{0_u(A)0_s(B_2)} \cdot \frac{0_s(y_b)}{0_s(x_b)}$$

$$= \frac{0_u(y_t-A)0_s(y_p)0_s(B_1)0_s(y_b)}{0_s(B_1)0_s(B_2)0_s(x_b)} \qquad \text{<3.65>}$$

Finally, $d_{r_n}(s)$ must divide $d_{c_n}(s)$, so that

$$\frac{d_{c_n}(s)}{d_{r_n}(s)} = \frac{0_u(y_t-A)0_s(y_p)0_s(B_1)0_s(y_b)0_s(n-M+1)}{0_s(B_1)0_s(B_2)0_s(x_b)} \qquad \text{<3.66>}$$

must be stable. Note that the denominator contains only LHP poles, since it is the product of stable polynomials. Therefore, the quotient is stable iff the degree of the denominator is equal to or greater than the degree of the numerator. This requires that

$$B_1+B_2+x_b \geq n-M+1+y_t-A+y_p+B_1+y_b$$

or, cancelling like terms and noting that $y_b = x_b$,

$$B_2 \geq n-M-1+2+Y_t-A+y_p$$

or

$$M+1 \geq y_t+y_p-A-B_2+n+2 = B_3 \qquad \text{<3.67>}$$

Then analyzing <3.67> further,

$$B_3 = y_t+y_p-A+n+2-B_2$$

$$= y_t+y_p-A+n+2-\max\{x_p-A+\max\{y_t,n+2-c+x_t\},y_t-A+y_p\}$$

$$= (y_t+y_p-A+n+2)+\min\{-x_p+A-\max\{y_t,n+2-c+x_t\},-y_t+A-y_p\}$$

since $-\max\{a,b\} = \min\{-a,-b\}$. Then, adding the term inside the

parenthesis to both sides inside the min function,

$$B_3 = \min\{y_t+y_p-A+n+2-x_p+A-\max\{y_t,n+2-c+x_t\}, \ y_t+y_p-A+n+2-y_t+A-y_p\}$$

$$= \min\{y_t+y_p+n+2-x_p-\max\{y_t,n+2-c+x_t\},n+2\}$$

$$= \min\{(y_t+y_p+n+2-x_p)+\min\{-y_t,-n-2+c-x_t\},n+2\}$$

$$= \min\{\min\{y_t+y_p+n+2-x_p-y_t,y_t+y_p+n+2-x_p-n-2+c-x_t\},n+2\}$$

$$= \min\{\min\{y_p-x_p+n+2,y_t+c-x_t+y_p-x_p\},n+2\}. \qquad \text{<3.68>}$$

The first quantity in the interior min function is y_p-x_p+n+2. But, according to Equation 3.55, $x_p \leq y_p$. Therefore,

$$y_p-x_p+n+2 \geq n+2$$

so <3.68> can be further reduced to

$$B_3 = \min\{n+2,y_t+c-x_t+y_p-x_p\}$$

$$M+1 \geq \min\{n+2,y_t+c-x_t+y_p-x_p\} \qquad \text{<3.69>}$$

Thus, in order for $d_{r_n}(s)$ to divide $d_{c_n}(s)$, the inequality in Equation 3.69 must be met. But, $M \leq n$, so $M+1$ cannot be equal to or greater than $n+2$. Therefore, the inequality must relate to the second term within the min function. Also, noting that

$$y_t+c-x_t = r\{t(s)\}$$

and

$$y_p-x_p - r\{p(s)\}$$

we can finally state that the necessary and sufficient condition is that

$$m+1 \geq r\{t(s)\} + r\{p(s)\} \qquad \text{<3.70>}$$

The above inequality is based on $d_{t_n}(s)$ according to Equation 3.57,

leaving the inequality based on $d_{t_n}(s)$ according to <3.58> yet to be proved. This proof is similar to the one just given, and is also lengthy, so in order to condense it many of the comments between steps will be omitted. First, however, it will prove useful to modify <3.58> so that

$$d_{t_n}(s) = \frac{0_u(c-n-2+y_t)}{0_s(\max\{x_t, c-n-2+y_t\})} \; ; \quad c-n-2 > 0 \qquad <3.71>$$

Then,

$$a_n(s) = \frac{n_p(s)}{d_{t_n}(s)} = \frac{0_u(x_p)}{0_s(y_p)} \cdot \frac{0_s(\max\{x_t, c-n-2+y_t\})}{0_u(c-n-2+y_t)}$$

$$= \frac{0_u(x_p-A')0_s(\max\{x_t, c-n-2+y_t\})}{0_u(c-n-2+y_t-A')0_s(y_p)} \qquad <3.72>$$

$$n_{a_n}(s) = \frac{0_u(x_p-A')0_s(B_4)}{0_s(B_5)} \qquad <3.73>$$

where

$$B_4 = \max\{x_t, c-n-2+y_t\}$$

$$B_5 = \max\{x_p-A'+B_4, c-n-2+y_t-A'+y_p\} \qquad <3.74>$$

so

$$d_{t_n}(s) = \frac{0_u(c-n-2+y_t)}{0_s(B_4)} \qquad <3.75>$$

Then

$$m_{a_n}(s) = \frac{n_p(s)}{n_{a_n}(s)} = \frac{0_u(x_p)}{0_s(y_p)} \cdot \frac{0_s(B_5)}{0_u(x_p-A')0_s(B_4)} = \frac{0_u(A')0_s(B_5)}{0_s(y_p)0_s(B_4)} \qquad <3.76>$$

$$m_{b_n}(s) = \frac{O_s(x_b)}{O_s(y_b)} \quad ; \quad x_b = y_b$$

$$d_{c_n}(s) = \frac{d_{t_n}(s)}{m_{a_n}(s)m_{b_n}(s)}$$

$$= \frac{O_u(c-n-2+y_t)}{O_s(B_4)} \cdot \frac{O_s(y_p)O_s(B_4)}{O_u(A')O_s(B_5)} \cdot \frac{O_s(y_b)}{O_s(x_b)}$$

$$= \frac{O_u(c-n-2+y_t-A')O_s(y_p)O_s(y_b)}{O_s(B_5)O_s(x_b)} \qquad <3.77>$$

Finally

$$\frac{d_{c_n}(s)}{d_{r_n}(s)} = \frac{O_u(c-n-2+y_t-A')O_s(y_p)O_s(y_b)}{O_s(B_5)O_s(x_b)} \cdot \frac{O_s(n-M+1)}{1} \qquad <3.78>$$

Again, <3.78> only requires

$$B_5+x_b \geq c-n-2+y_t-A'+y_p+y_b+n-M-1+2$$

or

$$M+1 \geq c+y_t+y_p+y_b-A'-x_b-B_5 = B_6 \qquad <3.79>$$

Then

$$B_6 = c+y_t+y_p+y_b-A'-x_b-\max\{x_p-A'+B_4, c-n-2+y_t-A'+y_p\}$$

$$= c+y_t+y_p+y_b-A'-x_b+\min\{-x_p+A'-B_4, -c+n+2-y_t+A'-y_p\}$$

$$= \min\{c+y_t+y_p+y_b-A'-x_b-x_p+A'-B_4, c+y_t+y_p+y_b-A'-x_b-c+n+2-y_t+A'-y_p\}$$

$$B_6 = \min\{c+y_t+y_p-x_p-B4, n+2\}$$

Which again requires that

$$B_6 = c+y_t+y_p-x_p-B_4 \qquad <3.80>$$

for a solution to exist. Then,

$$B_6 = c+y_t+y_p-x_p-\max\{x_t,c-n-2+y_t\}$$

$$= c+y_t+y_p-x_p+\min\{-x_t,-c+n+2-y_t\}$$

$$= \min\{c+y_t+y_p-x_p-x_t,c+y_t+y_p-x_p-c+n+2-y_t\}$$

$$= \min\{y_t+c-x_t+y_p-x_p,y_p-x_p+n+2\}$$

Also, $y_p \geq x_p$, and $n \geq M$, so $M+1$ can not be greater than y_p-x_p+n+2.

Thus, stability requires that

$$M+1 \geq (y_t+c-x_t) + (y_p-x_p)$$

Finally,

$$M+1 \geq r\{t(s)\} + r\{p(s)\} \tag{3.81}$$

Equations 3.70 and 3.81 are equivalent to Equation 3.51, thus proving Lemma 3.2.

EOP

If the conditions are met in Lemma 3.1 and Lemma 3.2, then a compensator exists that will stabilize the feedback loop and meet the initial n-th derivative condition. At this point, however, there is no guarantee that such a compensator will meet all other constraints also, which is the final goal. Thus, a complete parameterization meeting the n-th derivative constraint will now be derived. Then this parameterization will be analyzed to see when it will also meet the constraints $\{y_0,\ldots,y_{n-1}\}$.

Lemma 3.3

For the system in Figure 4, described by Equation 3.12, iff the

conditions $r\{t(s)\} + r\{p(s)\} \le M+1$, and $(d_{t_n}(s), d_p(s)) = 1$ are met, then a compensator exists that stabilizes both $w(s)$ and $z_n(s)$. In this case, $c(s)$ is described by Equation 3.8, where the complete parameterization for $w(s)$ is

$$w(s) = w_n(s) = (u_p(s)n_{a_n}(s)n_{t_n}(s) - n_{r_n}(s)\frac{d_{a_n}(s)}{d_{r_n}(s)})u_{s_n}(s) + d_{a_n}(s)e(s) \quad <3.82>$$

whenever $w(s)n_p(s) + v_p(s)$ is not equal to zero. The function $e(s)$ is arbitrarily stable, $a_n(s)$ is per Equation 3.19, and $u_{s_n}(s)$ (and $v_{s_n}(s)$) is stable, and satisfies

$$u_{s_n}(s)n_p(s)d_p(s)n_{t_n}(s) + v_{s_n}(s)d_{a_n}(s) = 1 \quad <3.83>$$

With $w_n(s)$ described per Equation 3.82

$$z_n(s) = (u_p(s)n_{a_n}(s)n_{t_n}(s) - n_{r_n}(s)\frac{d_{a_n}(s)}{d_{r_n}(s)})\, v_{s_n}(s)$$

$$-n_{a_n}(s)d_p(s)n_{t_n}(s)e(s) \quad <3.84>$$

Proof

To admit stable solutions, Property 3.10 requires stable $g_n(s)$, $k_n(s)$, and $x_n(s)$. By Lemma 3.1, $(d_{t_n}(s), d_p(s)) = 1$ is necessary and sufficient to stabilize $k_n(s)$. By Lemma 3.2, $r\{t(s)\} + r\{p(s)\} \le M+1$ is N & S to stabilize $g_n(s)$ and $x_n(s)$, by guaranteeing that $(d_{r_n} | d_{c_n})$. Thus, the conditions are justified.

Assume these conditions are met. Then, Equation 3.12 has been modified to Equation 3.36 (omitting the variable s and subscript n during

manipulation),

$$zd_c + wn_an_bn_t = \frac{u_pn_pn_t - x}{m_am_b}$$

$$= \frac{u_pn_pn_t - \dfrac{n_rd_t}{d_r}}{m_am_b} \qquad <3.85>$$

Also, $b_n(s)$ was defined by

$$b_n(s) = \frac{d_p(s)}{d_{t_n}(s)} = \frac{n_{b_n}(s)}{d_{b_n}(s)}$$

But, $(d_p(s), d_{t_n}(s)) = 1$, so we can let

$$d_p(s) = n_{b_n}(s) \quad ; \quad m_{b_n}(s) = 1 \quad ; \quad d_{t_n}(s) = d_{b_n}(s) \qquad <3.86>$$

Then, by Equation 3.26,

$$d_{t_n}(s) = m_{a_n}(s)m_{b_n}(s)d_{c_n}(s)$$

$$= m_{a_n}(s)d_{c_n}(s)$$

But, by <3.21>,

$$d_{t_n}(s) = m_{a_n}(s)d_{a_n}(s)$$

so that

$$m_{a_n}(s)d_{a_n}(s) = m_{a_n}(s)d_{c_n}(s)$$

or

$$d_{a_n}(s) = d_{c_n}(s)$$

Then, by Property 3.9, there exist stable $u_{s_n}(s)$ and $v_{s_n}(s)$ such that

$$u_{s_n}(s)n_{a_n}(s)n_{b_n}(s)n_{t_b}(s) + v_{s_n}(s)d_{c_n}(s) = 1$$

This can be modified to

$$u_{s_n}(s)n_{a_n}(s)d_p(s)n_{t_n}(s) + v_{s_n}(s)d_{a_n}(s) = 1 \qquad \text{<3.87>}$$

Also, <3.85> can be modified as follows.

$$zd_c + wn_a n_b n_t = \frac{u_p n_p n_t - \dfrac{n_r d_t}{d_r}}{m_a m_b}$$

$$zd_a + wn_a d_p n_t = \frac{u_p n_p n_t - \dfrac{n_r d_t}{d_r}}{m_a}$$

$$= \frac{u_p n_p n_t}{m_a} - \frac{n_r d_t}{m_a d_r}$$

$$= \frac{u_p n_p n_t}{\dfrac{n_p}{n_a}} - \frac{n_r m_a d_a}{m_a d_r}$$

$$zd_a + wn_a d_p n_t = u_p n_a n_t - n_r \frac{d_a}{d_r} \qquad \text{<3.88>}$$

The right side is stable since $dr_n | d_{a_n}$ ($d_{a_n} = d_{c_n}$).

Thus to find particular solutions, multiply by Equation 3.87 as follows.

$$zd_a + wn_a d_p n_t = (u_p n_a n_t - n_r \frac{d_a}{d_r}) \cdot (v_s d_a + u_s n_a d_p n_t)$$

$$= \{(u_p n_a n_t - n_r \frac{d_a}{d_r})v_s\}d_a + \{(u_p n_a n_t - n_r \frac{d_a}{d_r})u_s\}n_a d_p n_t \qquad \text{<3.89>}$$

Particular solutions are then equal to

$$w_n^p(s) = (u_p(s)n_{a_n}(s)n_{t_n}(s) - n_{r_n}(s)\frac{d_{a_n}(s)}{d_{r_n}(s)})u_{s_n}(s)$$

$$z_n^p(s) = (u_p(s)n_{a_n}(s)n_{t_n}(s) - n_{r_n}(s)\frac{d_{a_n}(s)}{d_{r_n}(s)})v_{s_n}(s) \qquad <3.90>$$

To find the complete solution set, we also need the homogeneous
solutions. Typcially, this is accomplished by guessing the homogeneous
solutions that cause the left side of the equation (here, Equation 3.89)
to equal zero. Then, the guesses are tested to verify that they are in
fact the homogeneous solutions. Thus, first guess that

$$w^h = d_a e$$

$$z^h = -n_a d_p n_t e \qquad <3.91>$$

where $e(s)$ is arbitrarily stable, so that

$$z^h d_a + w^h n_a d_p n_t = (-n_a d_p n_t e)d_a + d_a e n_a d_p n_t = 0$$

as desired, verifying that <3.91> are homogeneous solutions. We must
also test that all homogeneous solutions are of the form of <3.91>. To
do this, assume that $\underline{w}_n^h(s)$ and $\underline{z}_n^h(s)$ are stable, and satisfy

$$\underline{z}^h d_a + \underline{w}^h n_a d_p n_t = 0 \qquad <3.92>$$

and define the function $\underline{e}(s)$ by

$$\underline{e} = \frac{\underline{w}^h}{d_a} \qquad <3.93>$$

Clearly,

$$\underline{w}^h = \underline{e}d_a \qquad <3.94>$$

and it follows from <3.89> that

$$\underline{z}^h = \frac{-\underline{e}d_a n_a d_p n_t}{d_a} = -n_a d_p n_t \underline{e} \qquad \text{<3.95>}$$

showing that $\underline{w}^h$ and $\underline{z}^h$ have the form of Equation 3.91. Finally, to insure the completeness of the solution set, $\underline{e}(s)$ must be shown to be stable. To do this, the coprimeness properties of $n_{a_n}(s)$ and $d_{a_n}(s)$, and of $n_{t_n}(s)$ and $d_{t_n}(s)$ are needed, as follows.

$$\underline{e} = \frac{\underline{w}^h}{d_a}(u_a n_a + v_a d_a)$$

$$= w^h v_a + \frac{\underline{w}_h u_a n_a}{d_a}(u_t n_t + v_t d_t)$$

$$= w^h v_a + \frac{\underline{w}_h u_a n_a u_t n_t}{d_a} + \frac{\underline{w}^h u_a n_a v_t d_t}{d_a}$$

Invoking $m_a d_a = d_t$ and $(d_{t_n}, dp) = 1$,

$$\underline{e} = \underline{w}^h v_a + \frac{\underline{w}^h u_a n_a v_t m_a d_a}{d_a} + \frac{\underline{w}_h u_a n_a u_t n_t}{d_a}(u_{tp}d_t + v_{tp}d_p)$$

$$\underline{e} = \underline{w}^h v_a + \underline{w}^h u_a n_a v_t m_a + \underline{w}_h u_a n_a u_t n_t u_{t_p} m_a + \frac{\underline{w}_h u_a n_a u_t n_t v_{tp}d_p}{d_a} \qquad \text{<3.96>}$$

Also, by modifying Equation 3.92 yields

$$\underline{z}^h = \frac{-\underline{w}^h n_a d_p n_t}{d_a} \qquad \text{<3.97>}$$

Thus, substituting <3.97> into <3.96> leads to

$$\underline{e} = \underline{w}^h v_a + \underline{w}^h u_a n_a v_t m_a + \underline{w}_h n_a u_t n_t u_{tp} m_a - \underline{z}^h u_a u_t v_{tp} \qquad \text{<3.98>}$$

which proves $\underline{e}(s)$ is stable since it is the product and sum of stable functions. Therefore, since the complete solution set is the sum of the particular solution and the homogeneous solution, Equation 3.82 describes the general solution set for $w(s)$, and $z_n(s)$ is described by <3.84>.

To complete the proof, we must show that at least one stable $e(s)$ leads to a $w(s)$ such that $w(s)n_p(s) + v_p(s)$ is not identically equal to zero, in which case since this is the denominator of the compensator, the resulting compensator function would be undefined. Thus, again omitting the variable s and subscript n,

$$wn_p + v_p = \{(u_p n_a n_t - n_r \frac{d_a}{d_r})u_s + d_a e\}n_p + v_p$$

$$= \{(u_p n_a n_t - n_r \frac{d_a}{d_r})u_s n_p + v_p\} + ed_a n_p \qquad <3.99>$$

The function $d_{a_n}(s)$ is not zero since it is the denominator of $a_n(s)$. Also, if $n_p(s)$ is not zero, then $w(s)n_p(s) + v_p(s)$ is a non-trivial function of $e(s)$, and is therefore not identically zero for all $e(s)$. If $n_p(s)$ is equal to zero, then <3.7> implies that $v_p(s)$ is miniphase, and therefore can not be equal to zero. Thus, in this case $w(s)n_p(s) + v_p(s) = v_p(s)$ is not equal to zero. Therefore, the proof is complete.

EOP

3.6 Simultaneous Derivative Constraints

Clearly, Lemma 3.3 could be used to find a compensator meeting all initial response constraints simultaneously by applying it to each and

every separate constraint, from which the resulting parameterization would be the intersecting parameterization set of all the individual parameterizations. This could require a great deal of computation since $t_n(s)$, $r_n(s)$, and the resulting computed functions such as $a_n(s)$ depend on n. This normally will not be required though. Usually (and possibly always) solving just the n-th derivative case also will guarantee solving the 0-th through (n-1)th cases. A set of necessary and sufficient conditions for this guarantee have not been found, but the following lemma proves that usually only the n-th derivative parameterization will need to be computed.

Lemma 3.4

Suppose the conditions are met to apply Lemma 3.3 for the n-th derivative initial response constraint, so that $w_n(s)$ is computed per Equation 3.82, resulting in $z_n(s)$ per Equation 3.84. Then any compensator based on Equations 3.82 and 3.8 will stabilize not only $z_n(s)$, but also $z_0(s)$ through $z_{n-1}(s)$, whenever

1. The input $t(s)$ has fewer than three poles at $s = 0$.   <3.100>

or

2. If the above condition is not met, then   <3.101>

   {(#poles $t(s)$ at $s = 0$) - 3} < {# zeros $p(s)$ at $s = 0$}.

Proof

$z_n(s)$ is given by

$$z_n(s) = t_n(s)\{-w(s)n_p(s)d_p(s) + u_p(s)n_p(s)\} - r_n(s)$$

$$= s^{n+2}t(s)H_{v_2u_1}(s) - \sum_{i=0}^{n} s^{n-i+1}y_i \qquad <3.102>$$

Where $H_{v_2u_1}(s)$ is stable whenever $w(s)$ is stable, as is the case in Equation 3.82. Then, manipulating <3.102> leads to

$$\frac{z_n(s)}{s} = s^{n+1}t(s)H_{v_2u_1}(s) - \sum_{i=0}^{n} s^{n-i}y_i$$

$$\frac{z_n(s)}{s} + y_n = s^{n+1}t(s)H_{v_2u_1}(s) - \sum_{i=0}^{n-1} s^{n-i}y_i \qquad <3.103>$$

But, using the same set of constraints,

$$z_{n-1}(s) = s^{(n-1)+2}t(s)H_{v_2u_1}(s) - \sum_{i=0}^{n-1} s^{(n-1)-i+1}y_i$$

$$= s^{n+1}t(s)H_{v_2u_1}(s) - \sum_{i=0}^{n-1} s^{n-i}y_i \qquad <3.104>$$

Comparing <3.103> and <3.104> leads directly to

$$z_{n-1}(s) = \frac{z_n(s)}{s} + y_n \qquad <3.105>$$

Thus, since y_n is a constant, its sufficient but possibly not necessary to prove the $z_{n-1}(s)$ is stable by proving s divides z_n. Then, if $w(s)$ is given by <3.82>, $z_n(s)$ is given by <3.84> to be

$$z_n(s) = \{u_p(s)n_{a_n}(s)n_{t_n}(s) - n_{r_n}(s)\frac{d_{a_n}(s)}{d_{r_n}(s)}\} v_{s_n}(s)$$

$$- n_{a_n}(s)d_p(s) n_{t_n}(s)e(s)$$

where $dr_n(s)$ divides $d_{a_n}(s)$, so that $z_n(s)$ is stable. The above equation can be modified to

$$z_n(s) = n_{t_n}(s)n_{a_n}(s)\{u_p(s)v_{s_n}(s) - d_p(s)e(s)\}$$

$$- n_{r_n}(s)\{\frac{d_{a_n}(s)}{d_{r_n}(s)} v_{s_n}(s)\} \tag{<3.106>}$$

Clearly, $n_{r_n}(s)$ will always be divisable by s, because <2.39> describes $r_n(s)$ as a polynomial in s with no constant term. Thus, $z_n(s)$ will be divisable by s whenever either $n_{t_n}(s)$ or $n_{a_n}(s)$ are divisable by s (possibly other times too), since $z_n(s)$ will then be the sum of terms divisable by s.

Therefore, first consider $n_{t_n}(s)$, as follows.

Generally,

$$t(s) = \frac{0_u(x_t)}{s^c 0_u(y_t)} \tag{<3.107>}$$

is a representation for $t(s)$, where the s^c term represents the only poles or zeros at $s = 0$, so that c is any integer. This leads to

$$t_n(s) = s^{n+2}t(s) = \frac{s^{n+2-c}0_u(x_t)}{0_u(y_t)} \tag{<3.108>}$$

Obviously, $n_{t_n}(s)$ will be divisable by s iff

$$n+2-c > 0 \tag{<3.109>}$$

Now, if <3.109> holds for n, then $z_{n-1}(s)$ is stable. Then, this entire argument can be repeated for the case $z_{n-1}(s)$ is stable to show when $z_{n-2}(s)$ is stable, which will result in the inequality

$$(n-1)+2-c > 0 \tag{<3.110>}$$

Thus, to stabilize $z_0(s)$ through $z_{n-1}(s)$, the worst case inequality, which proves all $z_i(s)$, $i \leq 0 \leq n-1$, are stable, is the case when $n = 1$ and $(n-1) = 0$. The resulting inequality is

$$1+2-c > 0$$

or

$$c < 3 \qquad \qquad <3.111>$$

which is equivalent to saying $t(s)$ has fewer than three poles at $s = 0$. Therefore, $<3.100>$ is proved.

If $n_{t_n}(s)$ is not divisable by s, another condition that $z_n(s)$ is divisable by s is that $n_{a_n}(s)$ is divisable by s. Pursuing this case, first assume that $n_{t_n}(s)$ is not divisable by s, so that Equation 3.111 does not hold, but that

$$t_n(s) = \frac{0_u(x_t)}{s^{c-n-2}0_u(y_t)} \qquad \qquad <3.112>$$

where $c-n-2 \geq 0$ is a general expression for $t_n(s)$ based on Equation 3.107.

Thus,

$$d_{t_n}(s) = \frac{s^{c-n-2}0_u(y_t)}{0_s(\max\{x_t, c-n-2+y_t\})} \qquad \qquad <3.113>$$

Also, describe $p(s)$ generally by

$$p(s) = \frac{s^D 0_u(x_p)}{0_u(y_p)} \qquad \qquad <3.114>$$

where $D \geq 0$, and $0_u(y_p)$ may be divisable by s, so that D represents the

number of zeros the plant has at $s = 0$. Of course this leads to

$$n_p(s) = \frac{s^D 0_u(x_p)}{0_s(y_p)}$$ <3.115>

since $r\{p(s)\} \geq 0$ is a general assumption. This of course leads to, by definition,

$$a_n(s) = \frac{n_p(s)}{d_{t_n}(s)} = \frac{s^D 0_u(x_p)}{0_s(y_p)} \cdot \frac{0_s(\max\{x_t, c-n-2+y_t\})}{s^{c-n-2} 0_u(y_t)}$$ <3.116>

Obviously, $n_{a_n}(s)$ has $\max\{0, D-c+n+2\}$ zeros at s equal to 0. Thus, if

$$D-c+n+2 > 0$$ <3.117>

for all $n \geq 1$, then $n_{a_n}(s)$ will be divisable by s. Studying the worst case again of $n = 1$ leads to the general requirement that

$$D-c+1+2 > 0$$

or

$$c-3 < D$$ <3.118>

By our definitions, Equation 3.118 is equivalent to the statement of Equation 3.101, thus the proof is complete.

EOP

3.7 Initial Response Theorem and Parameterization

The exciting climax of this thesis is finally upon us! The groundwork has all been laid, so the Initial Response Theorem and the Initial Response Parameterization can now be given.

Theorem 3.2 - Initial Response Theorem (IRT)

For the system in Figure 2, assume that $r\{p(s)\} \geq 0$. Also assume

that

  1. The input $t(s)$ has fewer than three poles at $s = 0$.

or

  2. If the above condition is not met, then

  {(# poles $t(s)$ at $s = 0$)-3} < {# zeros $p(s)$ at $s = 0$}

Then a set of compensators $c(s)$ exists that will simultaneously stabilize

the feedback loop $H_{v_2 u_1}(s)$ (without RHP pole-zero cancellations between

$p(s)$ and $c(s)$), and meet the initial condition constraints

$$\{v_2^0(0^+),\ldots,v_2^j(0^+)\} = \{y_0,\ldots,y_j\} \text{ iff:}$$

  1. $r\{t(s)\} + r\{p(s)\} \leq M + 1$.

and

  2. $(d_{t_j}(s), d_p(s)) = 1$.

Where $y_i = 0$ for $i < M$, and $y_M \neq 0$.

Proof

The lemmas and properties derived earlier were dependent on the

assumption that $r\{p(s)\} \geq 0$. Then, Theorom 3.1 and Figure 4 lead to

Equation 3.12, which defines the complete set of compensators that

meet both feedback loop stability and the n-th derivative initial

response constraint are those which admit stable $w(s)$ and $z_n(s)$. Next,

Property 3.10 states that both $w(s)$ and $z_n(s)$ will be stable iff $g_n(s)$,

$k_n(s)$, and $x_n(s)$ admit stable solutions, as described per Equations 3.33,

3.34, and 3.35. Then, Lemma 3.1 and Lemma 3.2 give the necessary and

sufficient conditions to stabilize these three quantities to be

$$(d_{t_n}(s), dp(s)) = 1 \qquad\qquad <3.119>$$

and

$$r\{t(s)\} + r\{p(s)\} \leq M + 1 \qquad\qquad <3.120>$$

where $y_i = 0$ for $i < M$, and $y_M \neq 0$. Thus, <3.120> leads directly to condition 1 in the theorem, while <3.119> is dependent on the derivative constraint considered. Finally, Lemma 3.4 states that if Equations 3.100 or 3.101 are satisfied, then the n-th derivative solution also stabilizes all lower derivatives. Therefore, to get all constraints, let $n = j$ for the constraints $\{y_0,\ldots,y_j\}$ leading to the assumptions numbered 1 and 2 in Theorem 3.2, as well as $d_{t_n}(s) = d_{t_j}(s)$ in Equation 3.119, so that $(d_{t_j}(s), d_p(s)) = 1$ is required.
EOP

To reiterate, as the proof for Lemma 3.4 showed, <3.119> and <3.120> may not be necessary assumptions, but were shown to be sufficient. Also, obviously these pose very little restriction anyway, so that this point will not be labored further. Now the paramterization will be given for the compensators that achieve these constraints.

Theorem 3.3 - Initial Response Parameterization (IRP)

Given that the assumptions and conditions posed in the Initial Response Theorem are met for the constriants $\{v_2^0(0^+),\ldots v_2^j(0^+)\} = \{y_0,\ldots,y_j\}$. Then, let the plant be comprimely represented by

$$p(s) = \frac{n_p(s)}{d_p(s)}$$

so that stable $u_p(s)$ and $v_p(s)$ exist such that

$$u_p(s)n_p(s) + v_p(s)d_p(s) = 1$$

Also, define

$$t_j(s) = s^{j+2} t(s) = \frac{n_{t_j}(s)}{d_{t_j}(s)}$$

where $n_{t_j}(s)$ and $d_{t_j}(s)$ are coprime. Next, define

$$r_j(s) = \sum_{i=0}^{j} s^{j-i+1} y_i = \frac{n_{t_j}(s)}{d_{t_j}(s)}$$

where $n_{r_j}(s)$ and $d_{r_j}(s)$ are coprime, and let

$$a_j(s) = \frac{n_p(s)}{d_{t_j}(s)} = \frac{n_{a_j}(s)}{d_{a_j}(s)}$$

where $n_{a_j}(s)$ and $d_{a_j}(s)$ are coprime. Also, define stable $u_{s_j}(s)$ and $v_{s_j}(s)$ such that

$$u_{s_j}(s) n_{a_j}(s) d_p(s) n_{t_j}(s) + v_{s_j}(s) d_{a_j}(s) = 1$$

Then, the complete set of compensators that stabilize the feedback loop for the system in Figure 2 (without pole-zero cancellations between $p(s)$ and $c(s)$), and meet all $j + 1$ initial response constraint is given by

$$c(s) = \frac{-w(s) d_p(s) + u_p(s)}{w(s) n_p(s) + v_p(s)}$$

where $w(s)$ is stable, and given by

$$w(s) = \{ u_p(s) n_{a_j}(s) n_{t_j}(s) - n_{r_j}(s) \frac{d_{a_j}(s)}{d_{r_j}(s)} \} u_{s_j}(s) + d_{a_j}(s) e(s)$$

where e(s) is an arbitrarily stable function such that $w(s)n_p(s) + v_p(s)$ is not identically zero.

Proof

Theorom 3.2 states the assumptions and resulting N & S conditions for a parameterization to exist. Also, Lemma 3.3 gives that complete parameterization for w(s) leading to the n-th derivative constraint to be met. If we let n = j where j is the maximum order derivative initial constraint, then the assumptions in the IRT guarantee that for n = j in Equation 3.82, all constraints will simultaneously be met. Also, the proof for Lemma 3.3 showed that at least one e(s) exists so that $w(s)n_p(s) + v_p(s)$ is not zero, in which case the resulting compensator would be undefined, even though $z_n(s)$ and w(s) may be stable by Equation 3.12. Finally, c(s) was defined in Equation 3.8 exactly as stated in Theorom 3.3.

EOP

3.8 Summary

The IRP can be used to find the complete set of compensators that will meet any given initial response conditions that are within the bounds of the constraints and assumptions given in the IRT. This then gives the designer a tool that may help him to shape the transient response of a system as he desires. For example, designing for large positive initial values on the initial response derivatives should lead to a "faster" response than designing for smaller initial values on the same derivatives.

Theoretically, these theorems could be used to design precisely an entire system response. This can be seen by representing the frequency response of the entire system in its power series form, as follows.

$$H(s) = a_0 s^{-1} + a_1 s^{-2} + a_2 s^{-3} + a_3 s^{-4} + \ldots \tag{3.121}$$

Of course, this transforms to the time domain as

$$h(t) = \{a_0 + a_1 t + \frac{a_2}{2} t^2 + \frac{a_3}{6} t^3 + \ldots\} U(t)$$

$$= \sum_{i=0}^{\infty} \frac{a_i}{i!} t^i U(t) \tag{3.122}$$

It is easy to see that the initial i-th derivative is equal to a_i, i.e.,

$$h^i(0^+) = a_i \tag{3.123}$$

But, the IRP allows one to specify any and all $h^i(0^+)$ whenever the bounds of the IRT are not crossed. Thus, for any i, a_i can be specified, and looking back to <3.121>, one can see that any or all (theoroetically) of the power series for the transfer function can be specified!

CHAPTER IV

EXAMPLES

4.1 Introduction

In the previous chapter, the IRT and IRP were derived. Now,
several examples will be given which explore the usefulness of these
theoroms.

4.2 Introduction Example

In Section 1.2, an example problem was given. For that example,
Equation 1.2 and 1.8 were given to describe the set of compensators that
meet the desired constraints, but the equations were not derived. Thus,
the IRT and IRP will now be used to solve this problem. Figure 1
pictured the system in question, which is given again in Figure 5. For
the system, the constraints given were that the feedback loop be stable,
and $v_2(0^+)$ equal $3(= y_0)$.

For this system, $r\{p(s)\} = 1$, and the input has fewer than three
poles at $s = 0$ (it actually has none). Therefore, the assumptions
necessary for the IRT are met. Then, since $y_0 = 3$, M is equal to zero.
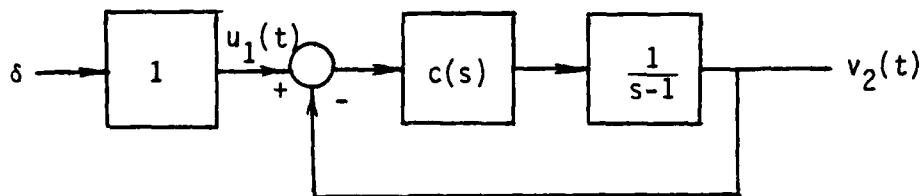Thus, the

Figure 5. Example From Introduction

first condition given in the IRT is met, i.e.,

$$r\{t(s)\} + r\{p(s)\} \leq M + 1 \quad ?$$

$$0 + 1 = 1 \leq 0 + 1 = 1 \quad \checkmark$$

Also, since $j = 0$, Equation 2.39 leads to

$$t_0(s) = s^{0+2}t(s) = s^2$$

This can be coprimely represented by

$$t_0(s) = \frac{\dfrac{s^2}{(s+1)^2}}{\dfrac{1}{(s+1)^2}} = \frac{n_{t_0}(s)}{d_{t_0}(s)} \qquad \text{<4.1>}$$

The plant can be coprimely represented by

$$p(s) = \frac{1}{s-1} = \frac{\dfrac{1}{s+1}}{\dfrac{s-1}{s+1}} = \frac{n_p(s)}{d_p(s)} \qquad \text{<4.2>}$$

Obviously, $(d_{t_0}(s), d_p(s)) = 1$, so the IRT dictates that a set of

compsensators exists that meets the desired constraints.

Now, moving on to the IRP, there must exist stable $u_p(s)$ and $v_p(s)$
such that

$$u_p(s)n_p(s) + v_p(s)d_p(s) = 1$$

or

$$u_p(s)\frac{1}{s+1} + v_p(s)\frac{s-1}{s+1} = 1 \qquad \text{<4.3>}$$

Obvious solutions to <4.3> are

$$u_p(s) = 2$$

and   <4.4>

$$v_p(s) = 1$$

Also,

$$r_0(s) = sy_0 = 3s = \frac{\frac{3s}{s+1}}{\frac{1}{s+1}} = \frac{n_{r_0}(s)}{d_{r_0}(s)} \qquad \text{<4.5>}$$

and

$$a_0(s) = \frac{n_p(s)}{d_{t_0}(s)} = \frac{\frac{1}{s+1}}{\frac{1}{(s+1)^2}} = s+1 = \frac{1}{\frac{1}{s+1}} = \frac{n_{a_0}(s)}{d_{a_0}(s)} \qquad \text{<4.6>}$$

There also must exist stable $u_{s_0}(s)$ and $v_{s_0}(s)$ such that

$$u_{s_0}(s)n_{a_0}(s)d_p(s)n_{t_0}(s) + v_{s_0}(s)d_{a_0}(s) = 1$$

or

$$u_{s_0}(s)\frac{s^2(s-1)}{(s+1)^3} + v_{s_0}\frac{1}{s+1} = 1 \qquad \text{<4.7>}$$

A little calculation yields adequate $u_{s_0}(s)$ and $v_{s_0}(s)$ to be

$$u_{s_0}(s) = 1$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ <4.8>

$$v_{s_0}(s) = \frac{4s^2+3s+1}{(s+1)^2}$$

The IRP then gives the complete set of adequate compensators to be

$$c(s) = \frac{-w(s)d_p(s) + u_p(s)}{w(s)n_p(s) + v_p(s)}$$

$$= \frac{-w(s)\frac{(s-1)}{(s+1)} + 2}{w(s)\frac{1}{(s+1)} + 1} \qquad \text{<4.9>}$$

where $w(s)$ will be described momentarily. Of course, <4.9> is exactly

equal to <1.2>. The IRP also describes w(s) by

$$w(s) = \{u_p(s)n_{a_0}(s)n_{t_0}(s) - n_{r_0}(s)\frac{d_{a_0}(s)}{d_{r_0}(s)}\}u_{s_0}(s) + d_{a_0}(s)e(s)$$

where e(s) is arbitrarily stable such that $w(s)n_p(s) + v_p(s)$ is not identically zero. Substituting in values for the current functions leads to

$$w(s) = \{2 \cdot 1 \cdot \frac{s^2}{(s+1)^2} - 3s\ \frac{1}{s+1}\}1 + \frac{1}{s+1}\ e(s)$$

which simplifies to

$$w(s) = \frac{-s^2 - 3s}{(s+1)^2} + \frac{1}{s+1}\ e(s) \qquad\qquad <4.10>$$

which is identical to <1.8>.

4.3 dc Motor Initial Response Problem

As a second example, consider for the plant a field-controlled dc motor. Reference (6) gives the transfer function for this type of motor as

$$p(s) = \frac{w}{v_f} = \frac{K}{RF}\ \frac{1}{(1+sT_f)(1+sT_m)} \qquad\qquad <4.11>$$

where the parameters are defined as follows.

  w = velocity - rad/sec

  v_f = input voltage - volts

  K = motor constant - lb-ft/field amperes

  R = field resistance - ohms

  F = total viscous friction - lb-ft/rad/sec

$$T_f = \text{field time constant} = \frac{L_f}{R_f} - \text{henry/ohm}$$

$$T_m = \text{mechanical time constant} - \text{rad-sec}$$

Also, assume the motor is powered by a step input equal to

$$t(s) = \frac{v}{s} \qquad \qquad \text{<4.12>}$$

and that the motor is controlled in a feedback system as in Figure 6.

Now analyzing this system using the IRT, we find that $r\{p(s)\} = 2 \geq 0$, and $t(s)$ has only one zero at $s = 0$, so the assumptions of the IRT are met.
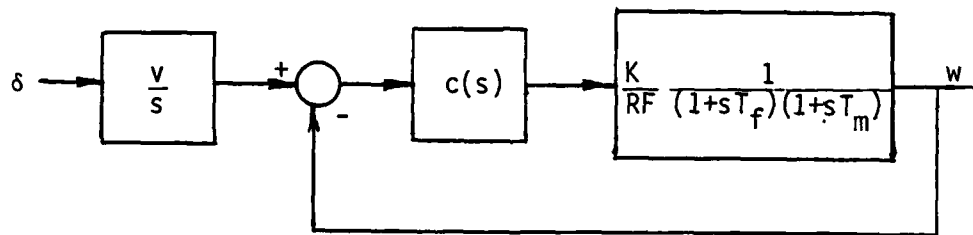

Figure 6. dc Motor Example

Then, $r\{t(s)\} = 1$, so

$$r\{t(s)\} + r\{p(s)\} = 1+2 = 3 \leq M+1$$

or

$$M \geq 2 \qquad \qquad \text{<4.13>}$$

thus demonstrating that any $c(s)$ will lead to $w(0^+)$ and $w^1(0^+)$ equal zero. It is also obvious that $(d_{t_j}(s), d_p(s)) = 1$ for any j. Therefore, a compensator parameterization can be found for any set of initial response constraints as long as $w(0^+)$ and $w^1(0^+)$ equal zero.

Intuition dictates that controlling even just the value of $w^2(0^+)$ should give some control over the transient response. Thus, this theory will be tested by requiring $\{w(0^+), w^1(0^+), w^2(0^+)\} = \{0,0,A\}$ where A is a real number. One would expect large values of A to lead to faster responses, with more overshoot than small values of A generally.

To that end, we apply the IRP with $j = 2$, and assume that neither T_f or T_m are equal to zero or one. Then, a coprime representation for $p(s)$ is

$$p(s) = \frac{\dfrac{\dfrac{K}{RF}}{(s+1)^2}}{\dfrac{(1+sT_f)(1+sT_m)}{(s+1)^2}} = \frac{n_p(s)}{d_p(s)} \qquad \text{<4.14>}$$

which leads to

$$u_p(s) = \frac{b_1 s + b_2}{s+1} \qquad \text{<4.15>}$$

and

$$v_p(s) = \frac{b_3 s + b_4}{s+1} \qquad \text{<4.16>}$$

where

$$b_3 = \frac{1}{T_f T_m} \qquad \text{<4.17>}$$

$$b_4 = \frac{3 - b_3(T_f + T_m)}{T_f T_m}$$

$$b_1 = \frac{3 - b_3 - (T_f + T_m)b_4}{\dfrac{K}{RF}}$$

$$b_2 = \frac{1-b_4}{\frac{K}{RF}}$$

Also, a coprime representation for $t_2(s)$ is

$$t_2(s) = s^4 \frac{v}{s} = \frac{\frac{vs^3}{(s+1)^3}}{\frac{1}{(s+1)^3}} = \frac{n_{t_2}(s)}{d_{t_2}(s)} \qquad <4.18>$$

and $r_2(s)$ is coprimely represented by

$$r_2(s) = \sum_{i=0}^{2} s^{2-i+1} w^i(0^+) = As = \frac{\frac{As}{(s+1)}}{\frac{1}{s+1}} = \frac{n_{r_2}(s)}{d_{r_2}(s)} \qquad <4.19>$$

Equations 4.14 and 4.18 lead to

$$a_2(s) = \frac{n_p(s)}{d_{t_2}(s)} = \frac{\frac{K}{RF}}{\frac{(s+1)^2}{\frac{1}{(s+1)^3}}} = \frac{K}{RF}(s+1)$$

which can be coprimely represented as

$$a_2(s) = \frac{\frac{K}{RF}}{\frac{1}{(s+1)}} = \frac{n_a(s)}{d_{a_2}(s)} \qquad <4.20>$$

The IVP also dictates that there are stable $u_{s_2}(s)$ and $v_{s_2}(s)$ such that

$$u_{s_2}(s)n_{a_2}(s)d_p(s)n_{t_2}(s) + v_{s_2}(s)d_{a_2}(s) = 1$$

and substituting

$$u_{s_2}(s) \frac{KV}{RF} \frac{s^3(1+sT_f)(1+sT_m)}{(s+1)^5} + v_{s_2}(s) \frac{1}{s+1} = 1 \qquad <4.21>$$

Solutions for $u_{s_2}(s)$ and $v_{s_2}(s)$ are

$$u_{s_2}(s) = \frac{c_1 s + c_2}{s+1} \qquad \qquad <4.22>$$

and

$$v_{s_2}(s) = \frac{c_3 s^4 + c_4 s^3 + c_5 s^2 + c_6 s + c_7}{(s+1)^5} \qquad \qquad <4.23>$$

where

$$c_1 = \frac{RF}{KV} \frac{1}{T_f T_m} \qquad \qquad <4.24>$$

and

$$c_2 = \frac{6 - \frac{KV}{RF} c_1 (T_f + T_m)}{\frac{KV}{RF} T_f T_m} \qquad \qquad <4.25>$$

Note that $c_3 - c_7$ can easily be found, but will not be written here since $v_{s_2}(s)$ is not needed to solve for $w(s)$. Now, the entire solution set for $w(s)$ is solved as follows.

$$w(s) = \{u_p(s) n_{a_2}(s) n_{t_2}(s) - r_2(s) d_{a_2}(s)\} u_{s_2}(s) + d_{a_2}(s) e(s)$$

$$= \{\frac{b_1 s + b_2}{s+1} \frac{K}{RF} \frac{vs^3}{(s+1)^3} - As \frac{1}{s+1}\} \frac{c_1 s + c_2}{s+1} + \frac{1}{s+1} e(s)$$

$$= \frac{d_1 s^5 + d_2 s^4 + d_3 s^3 + d_4 s^2 + d_5 s}{(s+1)^5} + \frac{1}{s+1} e(s) \qquad \qquad <4.26>$$

where

$$d_1 = \frac{1}{T_f T_m} \{b_1 - A \frac{RF}{KV}\} \quad ; \quad d_2 \text{ through } d_5 = \text{constants} \qquad \qquad <4.27>$$

and $e(s)$ is arbitrarily stable. Thus, the set of stabilizing compensators that cause $w^2(0^+) = A$ are given by

$$c(s) = \frac{-w(s)d_p(s)+u_p(s)}{w(s)n_p(s)+v_p(s)}$$

$$= \frac{-w(s)\dfrac{(1+sT_f)(1+sT_m)}{(s+1)^2} + \dfrac{b_1s+b_2}{s+1}}{w(s)\dfrac{\dfrac{K}{RF}}{(s+1)^2} + \dfrac{b_3s+b_4}{s+1}} \qquad <4.28>$$

where $\{b_1,b_2,b_3,b_4\}$ are given by <4.17>, and $w(s)$ is given by <4.26> and <4.27>. Note that since $e(s)$ is arbitrarily stable, but is multiplied by $1/(s+1)$, it can always be used to change any of the constants $\{d_2,d_3,d_4,d_5\}$ arbitrarily, but can not modify d_1. Thus, it is important only to know d_1.

One obvious choice for $e(s)$ is to set it so that

$$w(s) = \frac{d_1(s+1)^5}{(s+1)^5} = d_1 \qquad <4.29>$$

For example purposes, such a compensator was chosen to demonstrate how varying A affects the transient response of the system. Invoking Equation 3.10 leads to (where $h(s)$ represents the feedback loop)

$$t(s)h(s) = t(s)\{-w(s)n_p(s)d_p(s) + u_p(s)\ n_p(s)\}$$

$$= \frac{v}{s} \quad -d_1\frac{\dfrac{K}{RF}}{(s+1)^2}\ \frac{(1+sT_f)(1+sT_m)}{(s+1)^2} + \frac{(b_1s+b_2)}{(s+1)}\ \frac{\dfrac{K}{RF}}{(s+1)^2}$$

which after some manipulation reduces to

$$t(s)h(s) = \frac{As^2 + \frac{KV}{RF}\{b_1 + b_2 - d_1(T_f + T_m)\}s + \frac{KV}{RF}(b_2 - d_1)}{s(s+1)^4} \qquad <4.30>$$

A partial fractions expansion eventually helps lead to the time response

$$L^{-1}\{(t(s)h(s)\} = th(t)$$

$$= \{e_2 + e^{-t}(-e_2 - e_2 t + \frac{(A-e_2)}{2} t^2 + \frac{(e_1 - e_2 - A)}{6} t^3)\}U(s) \qquad <4.31>$$

where

$$e_1 = \frac{KV}{RF}(b_1 + b_2 - d_1(T_f + T_m)) \qquad <4.32>$$

and

$$e_2 = \frac{KV}{RF}(b_2 - d_1) \qquad <4.33>$$

Now, assume that the input $v = 100$ volts, and the motor has the following constants.

$K = 60$ lb-ft/amp

$R = 50$ ohm

$F = 2$ lb-ft/rad/sec   4.34

$T_f = .4$ henry/ohm

$T_m = 8.75$ rad-sec

These values were used with several values for A, and the resulting time responses were plotted in Figure 7 and Figure 8. Note that as expected, even though only $w^2(0^+)(=A)$ was directly controlled, increasing this produced faster responses and eventually increasing overshoot.
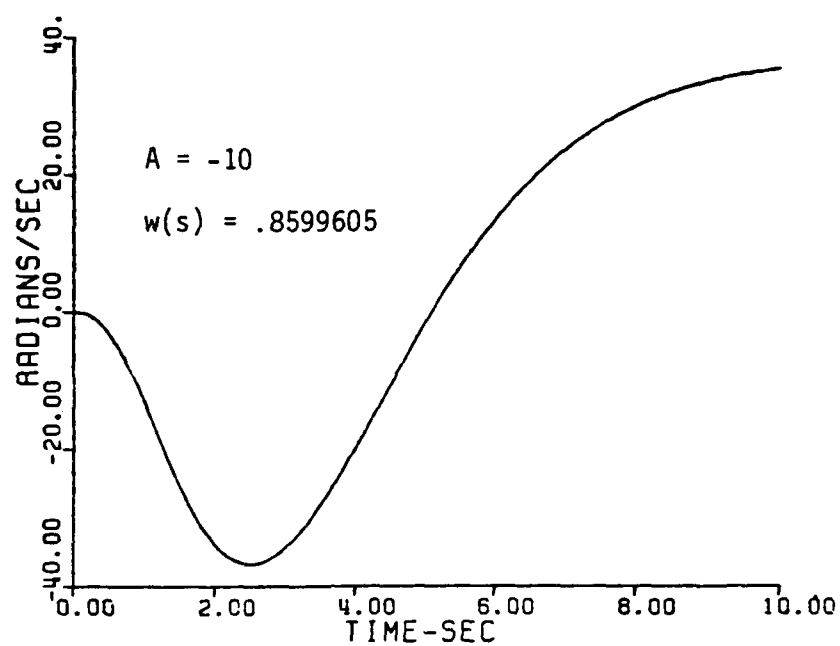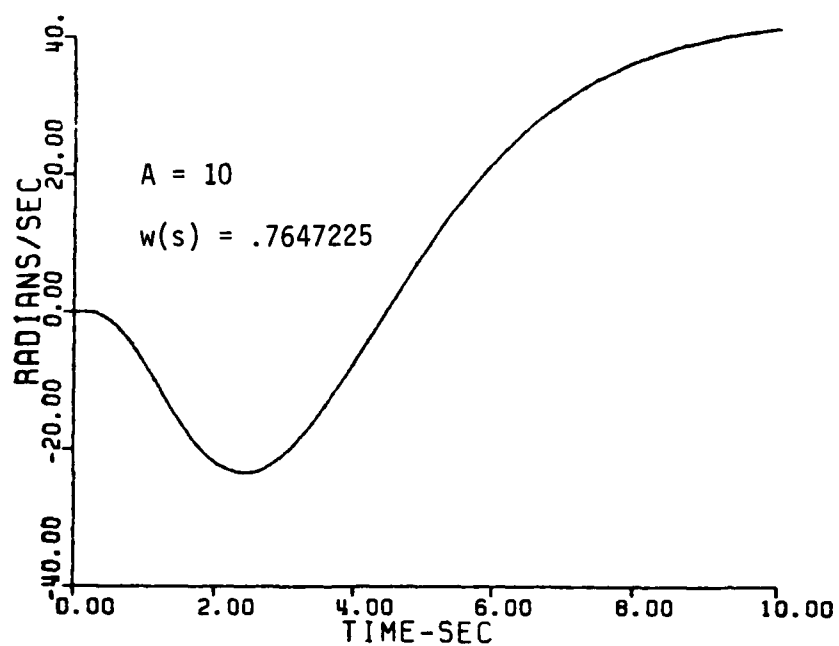
a. Response for A = -10

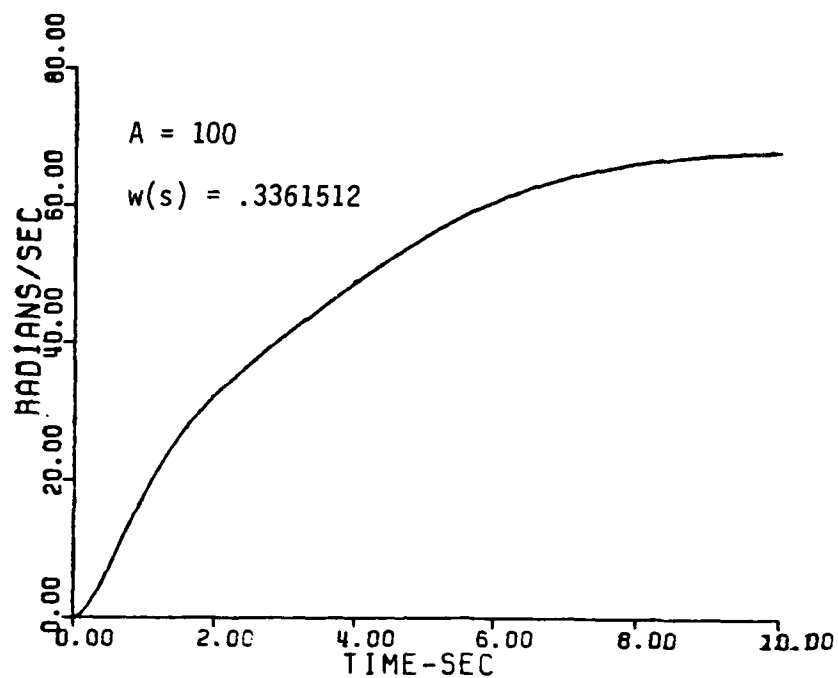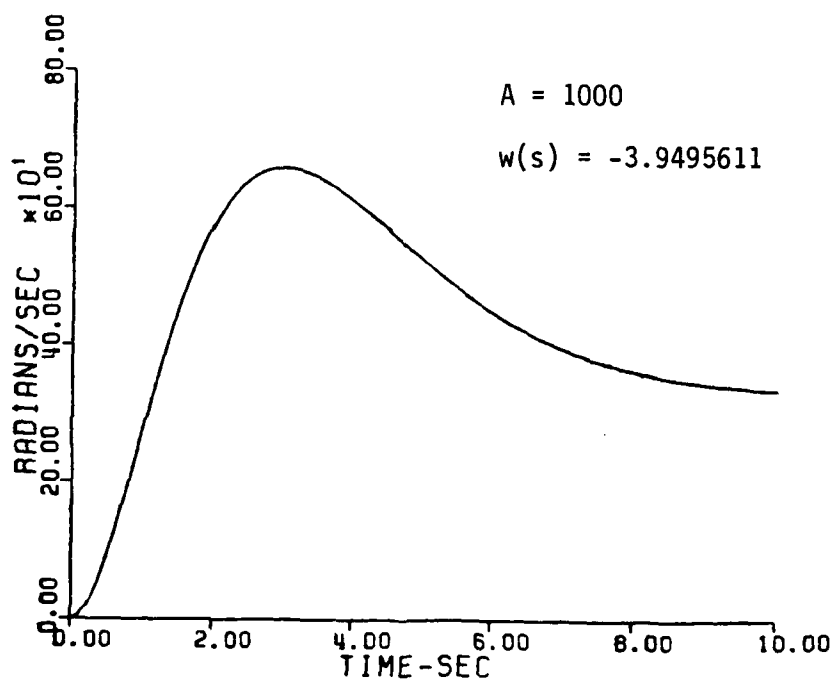

b. Response for A = 10

Figure 7.   Time Response Curves

a. Response for A = 100

b. Response for A = 1000

Figure 8. Time Response Curves

4.4 Combined Constraints Problem

The previous problem dealt with controlling only an initial value. Though the design technique yields feedback loop stability, there were no constraints on the asymptotic value of the motor velocity. Thus, as Figure 7 and Figure 8 show, using the previous technique alone can lead to a wide variety of final values. Of course, a common practical problem could arise in which both the initial response, and the asymptotic response must be simultaneously controlled. Although a general parameterization has not yet been derived, specific problems of this type can be treated now by invoking both the IVP, and the Tracking Theorem given in (10) and Appendix B. Compensators which meet both theorems can be found, and then analyzed to find the common parameterization.

As an example, consider the same motor in the previous section, with the same constants, and the same constraint $w^2(0^+) = A$. Thus, <4.26> describes $w(s)$, and <4.28> describes $c(s)$. Now, assume that the velocity of the motor shaft is to follow the input voltage v so that $w(\infty) = v$. This part of the problem can be solved by the Tracking Theorem iff the necessary and sufficient condition that $n_p(s)$ and $d_t(s)$ are coprime is met. Of course, this would not necessarily guarantee that both the IRP and the Tracking Theorem have any common solutions. Checking this conditions, $t(s)$ was given in <4.12>, and can be coprimely represented by

$$t(s) = \frac{\frac{v}{s+1}}{\frac{s}{s+1}} = \frac{n_t(s)}{d_t(s)} \qquad <4.35>$$

Equation 4.14 gives $n_p(s)$, which is obviously coprime with $d_t(s)$ above, so the condition is met. Thus, according to the Tracking Theorem, there exist stable $u_{pt}(s)$ and $v_{pt}(s)$ such that

$$u_{pt}(s)n_p(s) + v_{pt}(s)d_t(s) = 1 \qquad \text{<4.36>}$$

which leads to

$$u_{pt}(s) \frac{\frac{K}{RF}}{(s+1)^2} + v_{pt}(s) \frac{s}{s+1} = 1 \qquad \text{<4.37>}$$

Solutions for <4.37> are easily found to be

$$u_{pt}(s) = \frac{RF}{K}$$

and $\qquad$ <4.38>

$$v_{pt}(s) = \frac{s+2}{s+1}$$

Also

$$a'(s) = \frac{d_p(s)}{d_t(s)} = \frac{(1+sT_f)(1+sT_m)}{(s+1)^2} \cdot \frac{s+1}{s}$$

$$= \frac{(1+sT_f)(1+sT_m)}{s(s+1)} \qquad \text{<4.39>}$$

which can be coprimely represented by $a'(s) = n_{a'}(s)/d_{a'}(s)$, where

$$n_{a'}(s) = \frac{(1+sT_f)(1+sT_m)}{(s+1)^2}$$

and $\qquad$ <4.40>

$$d_{a'}(s) = \frac{s}{s+1}$$

Now, the Tracking Theorem gives the solution to be

$$c(s) = \frac{\{-w'(s)d_p(s) + u_p(s)\}}{\{w'(s)n_p(s) + v_p(s)\}} \qquad <4.41>$$

where

$$w'(s) = -u_{pt}(s)v_p(s) + e'(s)d_{a'}(s) \qquad <4.42>$$

with $e'(s)$ an arbitrarily stable function. Therefore, substituting into <4.42> gives

$$w'(s) = -\frac{RF}{K} \frac{b_3 s + b_4}{s+1} + e'(s) \frac{s}{s+1} \qquad <4.43>$$

Comparing <4.28> and <4.41>, one can conclude that simultaneous solutions to both the initial value problem and the tracking problem are those where $w(s) = w'(s)$. Thus, equating <4.26> and <4.43> leads to

$$\frac{d_1 s^5 + d_2 s^4 + d_3 s^3 + d_4 s^2 + d^5 s}{(s+1)^5} + \frac{1}{s+1} e(s)$$

$$= -\frac{RF}{K} \frac{b_3 s + b_4}{s+1} + e'(s) \frac{s}{s+1}$$

which can be rearranged to

$$\frac{-1}{s+1} e(s) + \frac{s}{s+1} e'(s) = \frac{d_1 s^5 + d_2 s^4 + d_3 s^3 + d_r s^2 + d_5 s}{(s+1)^5}$$

$$+ \frac{RF}{K} \frac{b_3 s + b_4}{s+1} \qquad <4.44>$$

The above equation can now be solved be differential equation methods, as follows. First, a particular solution can be found by recognizing the coprimeness of the functions multiplying $e(s)$ and $e'(s)$. Proceeding with this, we find that

$$-1 \cdot \frac{-1}{s+1} + 1 \cdot \frac{s}{s+1} = 1 \qquad <4.45>$$

Now, multiplying the right side of <4.44> by the left side of <4.45> leads to the particular solutions

$$e^p(s) = -\{\frac{d_1 s^5 + d_2 s^4 + d_3 s^3 + d_4 s^2 + d_5 s}{(s+1)^5} + \frac{RF}{K} \frac{b_3 s + b_4}{s+1}\} \qquad <4.46>$$

and

$$e'^p(s) = \{\frac{d_1 s^5 + d_2 s^4 + d_3 s^3 + d_4 s^2 + d_5 s}{(s+1)^5} + \frac{RF}{K} \frac{b_3 s + b_4}{s+1}\} \qquad <4.47>$$

Homogeneous solutions can be found by solving for

$$-\frac{1}{s+1} e^h(s) + \frac{s}{s+1} e'^h(s) = 0 \qquad <4.48>$$

Obvious solutions to the above equation are

$$e^h(s) = \frac{s}{s+1} k(s)$$

and

$$<4.49>$$

$$e'^h(s) = \frac{1}{s+1} k(S)$$

where $k(s)$ is arbitrarily stable. Normally, one would prove that <4.49> is in fact the complete homogeneous solution set, though this step will be skipped here. Now, the complete solutions are

$$e(s) = e^p(s) + e^h(s)$$

and

$$<4.50>$$

$$e'(s) = e'^p(s) + e'^h(s)$$

The compensator can now be represented by <4.41> and <4.42> or by <4.28> and <4.26>. Using <4.42>, we find that $w'(s)$ is given by

$$w'(s) = w(s) = -u_{pt}(s)v_p(s) + e'(s) \, d_{a'}(s)$$

$$= -\frac{RF}{K} \, \frac{b_3 s + b_4}{s+1} + \{(\frac{d_1 s^5 + d_2 s^4 + d_3 s^3 + d_4 s^2 + d_5 s}{(s+1)^5}$$

$$+ \frac{RF}{K} \, \frac{b_3 s + b_4}{s+1}) + k(s) \, \frac{1}{s+1}\} \, \frac{s}{s+1} \qquad <4.51>$$

As in the previous example, a particular $k(s)$ has been chosen for example purposes. In this case, $k(s)$ can be chosen so that the term inside the { } in <4.51> is equal to $d_1 + (RF/K)b_3$. Using this $k(s)$

$$w(s) = -\frac{RF}{K} \, \frac{b_3 s + b_4}{s+1} + \{d_1 + \frac{RF}{K} b_3\} \, \frac{s}{s+1}$$

$$= \frac{d_1 s - \frac{RF}{K} b_4}{s+1} \qquad <4.52>$$

Now, invoking <3.10> leads to (where $h(s)$ represents the feedback loop)

$$t(s)h(s) = \frac{v}{s} \{-\frac{d_1 s - \frac{RF}{K} b_4}{s+1} \, \frac{\frac{K}{RF}}{(s+1)^2} \, \frac{(1+sT_f)(1+sT_m)}{(s+1)^2}$$

$$+ \frac{b_1 s + b_2}{s+1} \, \frac{\frac{K}{RF}}{(s+1)^2}\}$$

which reduces to

$$t(s)h(s) = \frac{As^3 + q_2 s^2 + q_3 s + v}{s(s+1)^5} \qquad <4.53>$$

where

$$q_2 = \frac{KV}{RF} \{2b_1 + b_2 - d_1(T_f + T_m)\} + vb_4 T_f T_m$$

and   <4.54>

$$q_3 = \frac{KV}{RF} \{2b_2 + b_1 - d_1\} + vb_4(T_f + T_m)$$

Invoking the Final Value Theorem on <4.53>, we find that asymptotically w(t) is equal to v. Also, the Initial Value Theorem shows that $w^2(0^+) = A$. Thus, we do in fact have an adequate compensator. Now, a partial fractions expansion leads to the time domain response

$$L^{-1} \{t(s)h(s)\} = th(t)$$

$$= \{v + e^{-t}(-v - vt + (A-v)\frac{t^2}{2} + (q_2 - sA - v)\frac{t^3}{6}$$

$$+ (A - v + q_3 - q_2)\frac{t^4}{24} \} U(t)$$ <4.55>

The constants in Equation 4.34 were used for Equation 4.55. The resulting response curves were again plotted for several values fo A in Figure 9 and Figure 10. Note that the results are again as expected. Increasing A generally increases the speed of the response, and the amount of overshoot, while all of the curves have the steady state value w = 100 rad/sec.
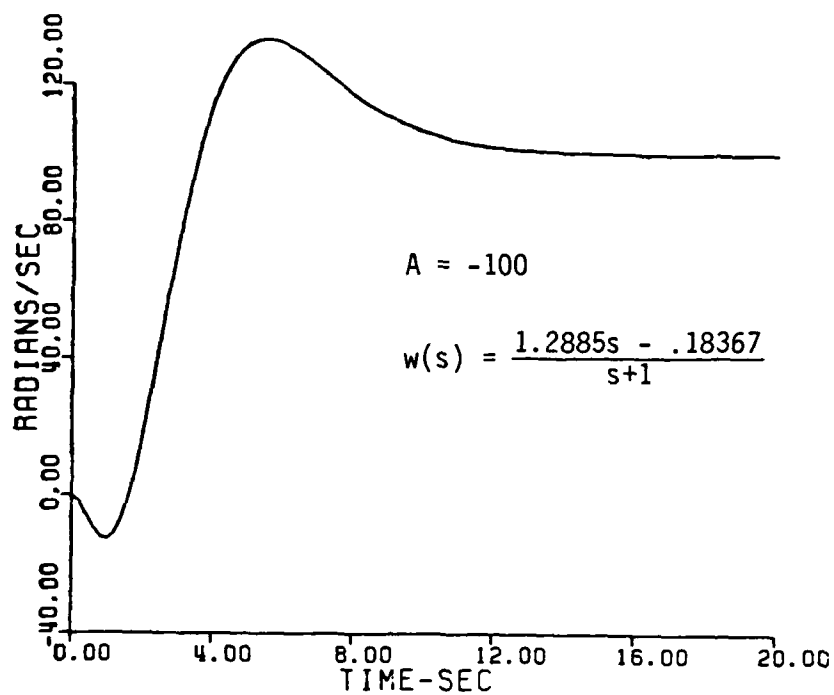
a. Response for A = -100 and v = 100
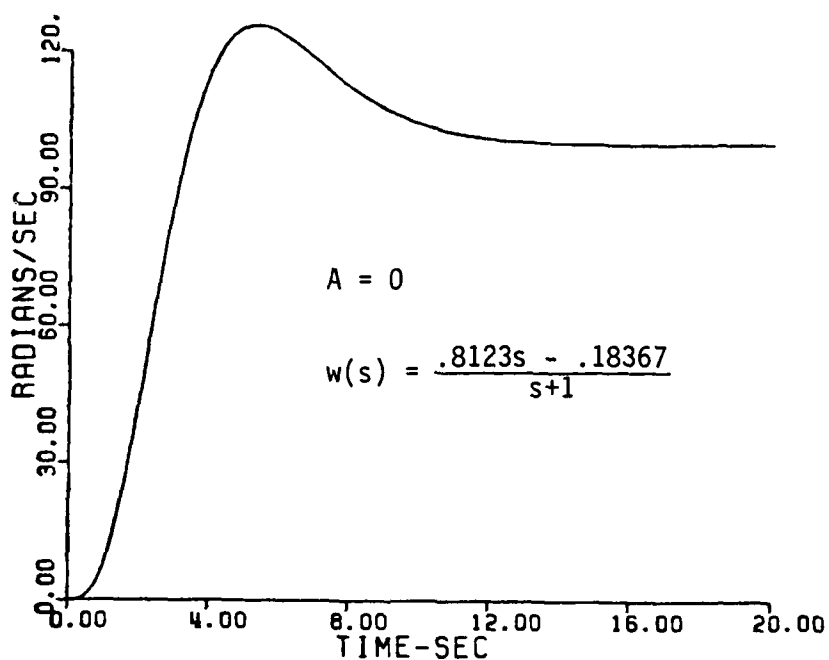
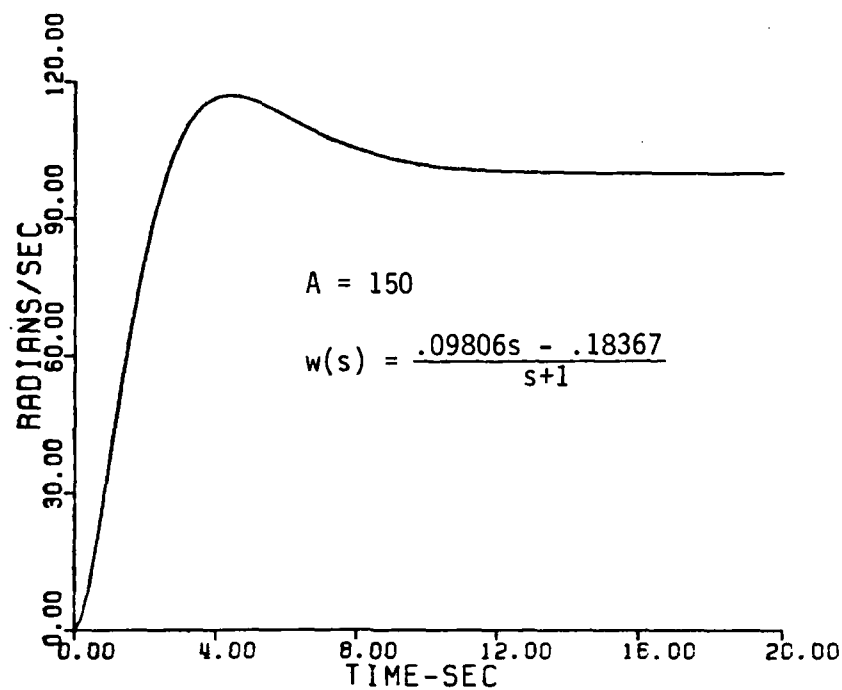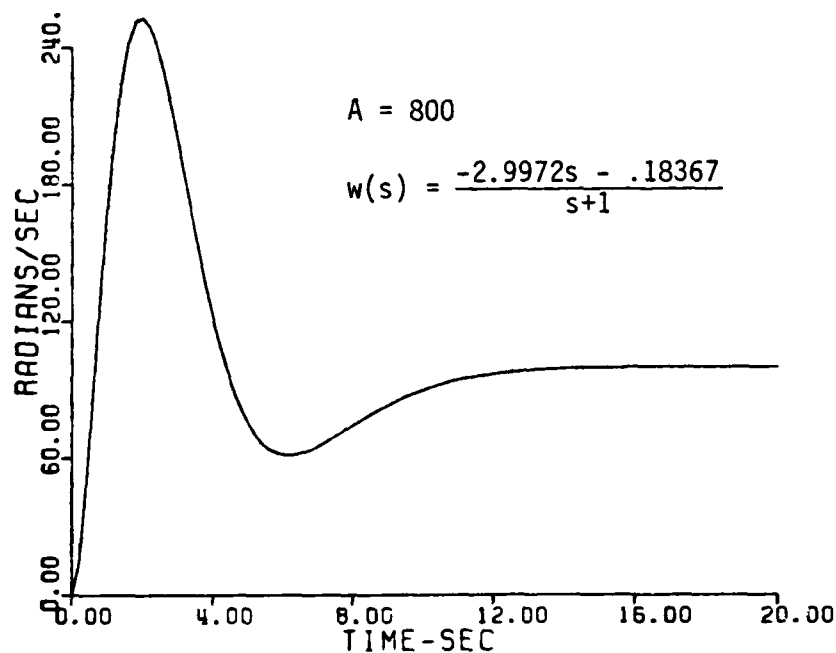b. Response for A = 0 and v = 100

Figure 9. Time Response Curves

a. Response for A = 150 and v = 100

b. Response for A = 800 and v = 100

Figure 10. Time Response Curves
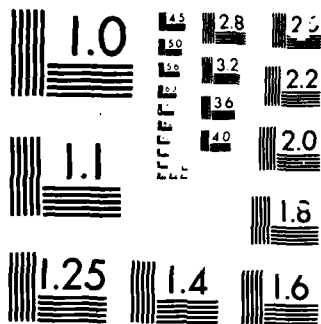
MICROCOPY CHART

CHAPTER V

RECOMMENDATIONS

The work in this thesis leaves the doors open for the study of several related topics. The following list contains recommendations of a few such problems which could be researched.

1. The problem considered here only utilizes unity gain in the feedback loop. An obvious extension is to consider the more general case with non-unity feedback gain.

2. To be truly useful for "real world" applications, this single-variate case should be extended to multivariate. Similar extensions have previously been handled for other parameterizations, for example in (1, 4, 7, 14, 16, 17, 18).

3. A third extension would be to develop a simultaneous Initial Response-Tracking parameterization, as well as other similar simultaneous parameterizations. These would be based on the IRT and IRP, and parameterizations such as those in (10, 11).

4. The examples in the previous chapter were designed to illustrate the usefulness of specifying initial response values. This technique should be explored and experimented with further so that perhaps a general design method would be derived. The power of specifying multiple initial response values particularly needs further exploration.

5. At the end of Chapter III, it was illustrated that specifying initial response values is equivalent to specifying the coefficients of the power series of the transfer function. To be completely

specified, of course, an infinite number of coefficients must be specified. However, the concept of approximately specifying the system transfer function by only specifying a finite number of coefficients could be pursued. This could lead to the ability to design for an entire specific response curve simply by applying the IRT and the IRP.

6. The IRT contains two assumptions based on Lemma 3.4 which guarantee that the parameterization for a particular initial response derivative will also meet the lower order derivative constraints. The assumptions are not very limiting, but it is possible that they are not required at all, or that less stringent assumptions could be used. This problem could be further researched.

7. The results in this thesis depend upon the system model, and no consideration has been given for the robustness necessary usually in the real world due to approximations, drift, etc. This robustness topic should be looked into further, as it has been for other parameterizations, as in (11).

REFERENCES

1. Antsaklis, P.J., & Pearson, J.B. Stabilization and Regulation in Linear Multivariate Systems. IEEE Trans. on Auto. Cont., 1978, AC-23, 928-930.

2. Bengtsson, G. Output Regulation and Internal Modes - A Frequency Domain Approach. Automatica, 1977, 13, 333-345.

3. Callier, F.M., & Desoer, C.A. Stabilization, Tracking and Disturbance Rejection in Linear Multivariable Distributed Systems. Proc. of the 17th IEEE Conf. on Decision and Control, Jan. 1979, 513-514.

4. Chang, L., & Pearson, J.B. Frequency Domain Synthesis of Multivariate Linear Regulators. IEEE Trans. on Auto. Cont., 1978, AC-23, 3-15.

5. Chua, O. M.S. Thesis, Texas Tech Univ., 1980.

6. Del Toro, V. Electromechanical Devices for Energy Conversion and Control Systems. New Jersey: Prentice-Hall, 1968.

7. Desoer, C.A., Liu, R.-w., Murray, J., & Saeks, R. Feedback System Design: the fractional representation approach to analysis and synthesis. IEEE Trans. on Auto. Cont., 1980, AC-25, 399-412.

8. Hayt, W.H., & Kemmerly, J.E. Engineering Circuit Analysis. New York: McGraw-Hill, 1978.

9. Saeks, R., & Murray, J. Feedback Systems Design: the tracking and disturbance rejection problems. IEEE Trans. on Auto. Cont., 1981, AC-26, 203-217.

10. Saeks, R., Murray, J., Chua, O., Karmokolias, C., & Iyer, A. Feedback System Design: The Single-Variate Case - Part I. Circuit Systems Signal Process, 1982, I-2, 137-169.

11. Saeks, R., Murray, J., Chua, O., Karmokolias, C., & Iyer, A. Feedback System Design: The Single-Variate Case - Part II. Circuits Systems Signal Process, 1983, II-1, 3-34.

12. Vidyasagar, M., Schneider, H., & Francis, B. Algebraic and Topological Aspects of Feedback System Stabilization. Tech Rpt. 80-90, Dept. of Elec. Engrg., Univ. of Waterloo, 1980.

13. Vidyasagar, M., & Viswanadham, M. Algebraic Design Techniques for Reliable Stabilization. Tech Rpt. 81-02, Dept. of Elec. Engrg, Univ. of Waterloo, 1980.

14. Wolovitch, W.A. Multivariate System Synthesis with Step Distrubance Rejection. IEEE Trans. on Auto. Cont., 1974, AC-19, 127-130.

15. Wolovitch, W.A., & Ferreira, P. Output Regulation and Tracking
     in Linear Multivariable Systems. <u>IEEE Trans. on Auto. Cont.</u>,
     1979, <u>AC-24</u>, 460-465.

16. Youla, D.C., Bongiorno, J.J., & Jabr, H.A. Modern Wiener - Hopf
     Design of Optimal Controllers, Part I. <u>IEEE Trans. on Auto. Cont.</u>,
     1976, <u>AC-21</u>, 3-15.

17. Youla, D.C., Bongiorno, J.J., & Jabr, H.A. Modern Wiener - Hopf
     Design of Optimal Controllers, Part II. <u>IEEE Trans. on Auto. Cont.</u>,
     1976, <u>AC-21</u>, 319-338.

18. Youla, D.C., Bongiorno, J.J., & Lu, C.N. Single-Loop Feedback
     Stabilization of Linear Multivariable Dynamical Plants. <u>Automatica</u>,
     1974, <u>10</u>, 159-173.

APPENDIX A

PROOFS FOR PROPERTIES

Several properties were given in Section 2.3 that will be proved in this appendix. The properties relate to the transformation described in Equation 2.17, which is repeated below.

$$\tilde{H}(\underline{s}) = \tilde{T}\{H(s)\} = H(\frac{1}{\underline{s}}) \qquad \text{<A.1>}$$

$$\tilde{T}^{-1}\{\tilde{H}(\underline{s})\} = \tilde{H}(\frac{1}{s})$$

Additive Property

$$\tilde{H}(\underline{s}) + \tilde{F}(\underline{s}) = \tilde{T}\{H(s) + F(s)\} \qquad \text{<A.2>}$$

Proof

Represent any two transfer functions, $H(s)$ and $F(s)$ as follows.

$$H(s) = \frac{(a_1 s + b_1) \dots (a_n s + b_n)}{(c_1 s + d_1) \dots (c_m s + d_m)}$$

$$\text{<A.3>}$$

$$F(s) = \frac{(e_1 s + f_1) \dots (e_p s + f_p)}{(g_1 s + j_1) \dots (g_k s + j_k)}$$

Of course, $\{a_i, b_i, c_i, d_i, e_i, f_i, g_i, j_i\}$ are complex numbers, and $\{n, m, p, k\}$ are positive integers. Then, according to <A.1>,

$$\tilde{H}(\underline{s}) = H(\frac{1}{\underline{s}}) = \frac{(\frac{a_1}{\underline{s}} + b_1) \dots (\frac{a_n}{\underline{s}} + b_n)}{(\frac{c_1}{\underline{s}} + d_1) \dots (\frac{c_m}{\underline{s}} + d_m)} \cdot \frac{\underline{s}^{n+m}}{\underline{s}^{n+m}}$$

$$= \frac{(a_1 + b_1 \underline{s}) \dots (a_n + b_n \underline{s}) \underline{s}^{m-n}}{(c_1 + d_1 \underline{s}) \dots (c_m + d_m \underline{s})} \qquad \text{<A.4>}$$

and

$$\tilde{F}(\underline{s}) = F(\frac{1}{\underline{s}}) = \frac{(\frac{e_1}{\underline{s}} + f_1)\ldots(\frac{e_p}{\underline{s}} + f_p)}{(\frac{g_1}{\underline{s}} + j_1)\ldots(\frac{g_k}{\underline{s}} + j_k)} \cdot \frac{\underline{s}^{p+k}}{\underline{s}^{p+k}}$$

$$= \frac{(e_1 + f_1\underline{s})\ldots(e_p + f_p\underline{s})\underline{s}^{k-p}}{(g_1 + j_1\underline{s})\ldots(g_k + j_k\underline{s})} \qquad \text{<A.5>}$$

Adding <A.4> and <A.5> yields

$$\tilde{H}(\underline{s}) + \tilde{F}(\underline{s}) = \frac{(a_1 + b_1\underline{s})\ldots(a_n + b_n\underline{s})\underline{s}^{m-n}}{(c_1 + d_1\underline{s})\ldots(c_m + d_m\underline{s})}$$

$$+ \frac{(e_1 + f_1\underline{s})\ldots(e_p + f_p\underline{s})\underline{s}^{k-p}}{(g_1 + j_1\underline{s})\ldots(g_k + j_k\underline{s})} \qquad \text{<A.6>}$$

Conversely, by <A.3>,

$$H(s) + F(s) = \frac{(a_1 s + b_1)\ldots(a_n s + b_n)}{(c_1 s + d_1)\ldots(c_m s + d_m)}$$

$$+ \frac{(d_1 s + f_1)\ldots(e_p s + f_p)}{(g_1 s + j_1)\ldots(g_k s + j_k)} \qquad \text{<A.7>}$$

Then, transforming the sum leads to

$$\tilde{T}\{H(s) + F(s)\} = \frac{(\frac{a_1}{\underline{s}} + b_1)\ldots(\frac{a_n}{\underline{s}} + b_n)\underline{s}^{n+m}}{(\frac{c_1}{\underline{s}} + d_1)\ldots(\frac{c_m}{\underline{s}} + d_m)\underline{s}^{n+m}}$$

$$+ \frac{(\frac{e_1}{\underline{s}} + f_1)\ldots(\frac{e_p}{\underline{s}} + f_p)\underline{s}^{p+k}}{(\frac{g_1}{\underline{s}} + j_1)\ldots(\frac{g_k}{\underline{s}} + j_k)\underline{s}^{p+k}}$$

$$\bar{T}\{H(s) + F(s)\} = \frac{(a_1 + b_1\underline{s})\ldots(a_n + b_n\underline{s})\underline{s}^{m-n}}{(c_1 + d_1\underline{s})\ldots(c_m + d_m\underline{s})}$$

$$+ \frac{(e_1 + f_1\underline{s})\ldots(e_p + f_p\underline{s})\underline{s}^{k-p}}{(g_1 + j_1\underline{s})\ldots(g_k + j_k\underline{s})} \qquad \text{<A.8>}$$

Equating <A.6> and <A.8> leads directly to

$$\bar{H}(\underline{s}) + \bar{F}(\underline{s}) = \bar{T}\{H(s) + F(s)\} \qquad \text{<A.9>}$$

EOP

Multiplicative Property

$$\bar{H}(\underline{s})\bar{F}(\underline{s}) = \bar{T}\{H(s)F(s)\} \qquad \text{<A.10>}$$

Proof

This proof proceeds exactly like the proof for the Additive Property, except that the product of H(s) and F(s) is considered rather than the sum. Hence, this proof will not be reiterated.

EOP

Inverse Transform Property

$$\bar{H}(\frac{1}{s}) = H(s) \qquad \text{<A.11>}$$

Proof

Again, represent H(s) per <A.3>, and the resulting $\bar{H}(\underline{s})$ per <A.4>. Then

$$\bar{H}(\frac{1}{s}) = \frac{(a_1 + \frac{b_1}{s})\ldots(a_n + \frac{b_n}{s})}{(c_1 + \frac{d_1}{s})\ldots(c_m + \frac{d_m}{s})s^{m-n}} \cdot \frac{s^{m+n}}{s^{m+n}}$$

$$\tilde{H}\left(\frac{1}{s}\right) = \frac{(a_1 s + b_1)\ldots(a_n s + b_n)s^m}{(c_1 s + d_1)\ldots(c_m s + d_m)s^{m-n}\, s^n}$$

$$= \frac{(a_1 s + b_1)\ldots(a_n s + b_n)}{(c_1 s + d_1)\ldots(c_m s + d_m)}$$

$$= H(s).$$

EOP

Transformation Stability Property

If $H(s)$ is stable (or unstable) in the s-domain, then so is $\tilde{H}(\underline{s})$ in the $\underline{s}$-domain, and visa versa.

Proof

Once more define $H(s)$ generally by Equation A.3 so that

$$H(s) = \frac{(a_1 s + b_1)\ldots(a_n s + b_n)}{(c_1 s + d_1)\ldots(c_m s + d_m)}$$

and the resulting $\tilde{H}(\underline{s})$ by <A.4>, thus

$$\tilde{H}(\underline{s}) = \frac{(a_1 + b_1\underline{s})\ldots(a_n + b_n\underline{s})\underline{s}^{m-n}}{(c_1 + d_1\underline{s})\ldots(c_m + d_m\underline{s})}$$

Then, first consider only stable $H(s)$. Under this condition, $n \leq m$, and the poles are all in the strict LHP. Of course, to meet this LHP requirement, the poles, p_i, which are located at

$$p_i = -\frac{d_i}{c_i} = |p_i| \angle \Theta_i \qquad \text{<A.12>}$$

must be such that

$$|p_i| > 0, \text{ but finite} \qquad \text{<A.13>}$$

and

$$\theta_i: \quad 90^0 < \theta_i \leq 180^0, \text{ or } -180^0 > \theta_i > -90^0 \qquad \text{<A.14>}$$

To meet Equation A.13 requires that $\{c_i, d_i\}$ contains no elements equal to zero, but no such requirement is on $\{a_i, b_i\}$. Therefore, if there are A elements in the set $\{b_i\}$, $1 \leq i \leq n$, equal to zero, then

$$r\{\tilde{H}(\underline{s})\} = m - (n-A) - (m-n) = A \geq 0 \qquad \text{<A.15>}$$

proving that $\tilde{H}(\underline{s})$ is proper, as required for stability. Also, $\tilde{H}(\underline{s})$ has its poles located at $\tilde{p}_i$, where

$$\tilde{p}_i = -\frac{c_i}{d_i} = \frac{1}{|p_i|} \angle -\theta = |\tilde{p}_i| \angle \tilde{\theta}_i \qquad \text{<A.16>}$$

Since $|p_i|$ is greater than zero and less than infinity, so is $|\tilde{p}_i|$. Also, since $\tilde{\theta}_i = -\theta_i$, then by <A.14>,

$$\tilde{\theta}_i: \quad 90^0 < \tilde{\theta}_i < 180^0, \text{ or } -180^0 \geq \tilde{\theta}_i > -90^0 \qquad \text{<A.17>}$$

so that $\tilde{\theta}_i$ is in the LHP. Therefore, $\tilde{H}(\underline{s})$ is stable if $H(s)$ is.

Consider now unstable $H(s)$. First consider the case where the transfer function is not proper, so that $n > m$. Analysis of <A.4> illustrates that this results in $(n-m)$ poles at $\underline{s} = 0$ for $\tilde{H}(\underline{s})$, which makes $\tilde{H}(\underline{s})$ unstable. Next, consider finite poles of $H(s)$ in the closed RHP. In this case

$$\theta_i: \quad -90^0 \leq \theta_i \leq 90^0 \qquad \text{<A.18>}$$

which by Equation A.16 results in $\tilde{\theta}_i$ in this same unstable region, since $\tilde{\theta}_i = -\theta_i$. Therefore, unstable $H(s)$ lead to unstable $\tilde{H}(\underline{s})$.

Conversely, the inverse transform $\tilde{H}(\frac{1}{s})$ is equal to $H(s)$ by the

Inverse Transform Property. This is symmetrical with the transformation of $\tilde{H}(\underline{s}) = H(\frac{1}{s})$. Therefore, stability (or unstability) of transformation for the inverse transform holds also, because of this symmetry property.

EOP

APPENDIX B

TRACKING THEOREM

Reference (10) states the following theorom. The proof can be found in that reference.

Tracking Theorem

Give p(s) there exists a compensator for the feedback system of Figure 2 which stabilizes the feedback loop and simultaneously causes it to track the impulse response of t(s) if and only if $n_p(s)$ and $d_t(s)$ are coprime. In this case let $u_{pt}(s)$ and $v_{pt}(s)$ be stable functions such that

$$u_{pt}(s)n_p(s) + v_{pt}(s)d_t(s) = 1$$

and let $a'(s) = n_{a'}(s)/d_{a'}(s)$ be a coprime fractional representation of $a'(s) = d_p(s)/d_t(s)$. Then the desired set of compensators take the form

$$c(s) = \frac{\{-w(s)d_p(s) + u_p(s)\}}{\{w(s)n_p(s) + v_p(s)\}}$$

where

$$w(s) = -u_{pt}(s)v_p(s)v_p(s) + e(s)d_{a'}(s)$$

with e(s) an arbitrarily stable function such that $w(s)n_p(s) + v_p(s)$ is not identically zero.

EOT

BIOGRAPHICAL SKETCH

Paul Jay Warden was born in Pontiac, Michigan, on July 22, 1960. In 1978 he graduated from Lakeland High School in Highland, Michigan. He subsequently entered General Motors Institute in Flint, Michigan, where he studied Electrical Engineering and was a cooperative student employee with Chevrolet Saginaw Manufacturing in Saginaw, Michigan. He received the degree BEE from GMI in 1983, and then worked as a project engineer in the Plant Engineering Department at Chevrolet Saginaw Manufacturing until October of 1983. In 1984 he attended the United States Air Force Officers' Training School in San Antonio, Texas, and was commissioned a 2nd Lt in May, 1984. In August, 1984, Paul Warden was assigned to Arizona State University in order to work on the MSEE degree, and in January of 1986 he will be reassigned to Cape Canaveral Air Force Station in Florida. He is a member of the Theta Xi Fraternity.

END
FILMED

6-86

DTIC